Question 4.

In a small time interval Δt the fish moves upward, from point A to point B, at a small distance $d = v\Delta t$. Since the glass wall is very thin we can assume that the rays leaving the aquarium refract as if there was water – air interface. The divergent rays undergoing one single refraction, as show in Fig. 4.1, form the first, virtual, image of the fish. The corresponding vertical displacement A_1B_1 of that image is equal to the distance d_1 between the optical axis a and the ray b_1, which leaves the aquarium parallel to a. Since distances d and d_1 are small compared to R we can use the small-angle approximation: $\sin\alpha \approx \tan\alpha \approx \alpha$ (rad). Thus we obtain:

(4.1) $d_1 \approx R\,\alpha;$

(4.2) $d \approx R\,\gamma;$

(4.3) $\alpha + \gamma = 2\beta;$

(4.4) $\alpha \approx n\beta.$

From equations (4.1) - (4.4) we find the vertical displacement of the first image in terms of d:

(4.5) $$d_1 = \frac{n}{2-n} d\,,$$

and respectively its velocity v_1 in terms of v:

(4.6) $$v_1 = \frac{n}{2-n} = 2v\,.$$

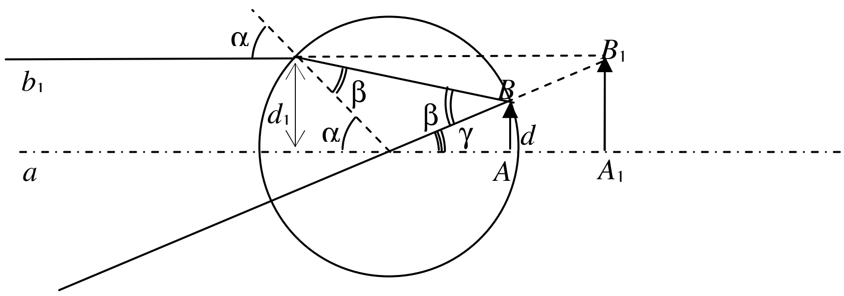

Fig. 4.1

The rays, which are first reflected by the mirror, and then are refracted twice at the walls of the aquarium form the second, real image (see Fig. 4.2). It can be considered as originating from the mirror image of the fish, which move along the line $A'B'$ at exactly the same distance d as the fish do.

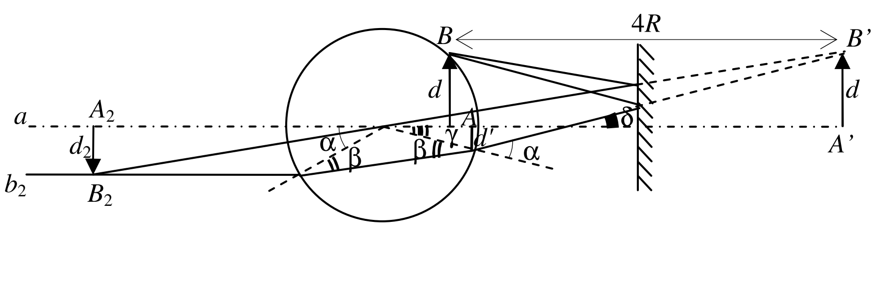

Fig. 4.2

The vertical displacement A_2B_2 of the second image is equal to the distance d_2 between the optical axis a and the ray b_2, which is parallel to a. Again, using the small-angle approximation we have:

(4.7) $$d' \approx 4R\delta - d,$$

(4.8) $$d_2 \approx R\alpha$$

Following the derivation of equation (4.5) we obtain:

(4.9) $$d_2 = \frac{n}{2-n} d' .$$

Now using the exact geometric relations:

(4.10) $$\delta = 2\alpha - 2\beta$$

and the Snell's law (4.4) in a small-angle limit, we finally express d_2 in terms of d:

(4.11) $$d_2 = \frac{n}{9n-10} d ,$$

and the velocity v_2 of the second image in terms of v:

(4.12) $$v_2 = \frac{n}{9n-10} v = \frac{2}{3} v .$$

The relative velocity of the two images is:

(4.13) $$\mathbf{v}_{\mathrm{rel}} = \mathbf{v}_1 - \mathbf{v}_2$$

in a vector form. Since vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are oppositely directed (one of the images moves upward, the other, downward) the magnitude of the relative velocity is:

(4.14) $$v_{\mathrm{rel}} = v_1 + v_2 = \frac{8}{3} v .$$