Solution Problem 3

The initial energy on the capacitor is:

$$W_o = \frac{1}{2} \cdot C_o U_o^2 = \frac{1}{2} \cdot \frac{Q_o^2}{C_o}, \text{ where } \quad C_o = \frac{\varepsilon_o H l}{d} \quad (1)$$

H is the height of the plates, l is the width of the capacitor's plates, and d is the distance between the plates.
When the plates contact the liquid's surface on the dielectric liquid is exerted a vertical force. The total electric charge remains constant and there is no energy transferred to the system from outside. The increase of the gravitational energy is compensated by the decrease of the electrical energy on the capacitor:

$$W_o = W_1 + W_2 \qquad (2)$$

$$W_1 = \frac{1}{2} \cdot \frac{Q_o^2}{C}, \quad W_2 = \frac{1}{2} \rho g h^2 l d \qquad (3)$$

$$C = C_1 + C_2 = \frac{\varepsilon_o \varepsilon_r h l}{d} + \frac{\varepsilon_o (H - h) l}{d} \qquad (4)$$

Introducing (3) and (4) in equation (2) it results:

$$(\varepsilon_r - 1) h^2 + Hh - \frac{E_o^2 \varepsilon_o H (\varepsilon_r - 1)}{\rho g} = 0$$

The solution is:

$$h_{1,2} = \frac{H}{2(\varepsilon_r - 1)} \cdot \left[-1 \pm \sqrt{1 \pm \frac{4 E_o^2 \varepsilon_o (\varepsilon_r - 1)^2}{\rho g H}} \right] \qquad (8)$$

Discussion: Only the positive solution has sense. Taking in account that H is much more grater than h we obtain the final result:

$$h \approx \frac{\varepsilon_o (\varepsilon_r - 1)}{\rho g} \cdot E_o^2$$