Problem 1.

1) During the rocket moving along the circular orbit its centripetal acceleration is created by moon gravity force:

$$G\frac{MM_M}{R^2} = \frac{Mv_0^2}{R},$$

where $R = R_M + h$ is the primary orbit radius, v_0 -the rocket velocity on the circular orbit:

$$v_0 = \sqrt{G\frac{M_M}{R}}$$

Since $g_M = G\frac{M_M}{R_M^2}$ it yields

$$v_0 = \sqrt{\frac{g_M R_M^2}{R}} = R_M\sqrt{\frac{g_M}{R_M + h}} \qquad (1)$$

The rocket velocity will remain perpendicular to the radius-vector OA after the braking engine sends tangential momentum to the rocket (Fig.1). The rocket should then move along the elliptical trajectory with the focus in the Moon's center.

Denoting the rocket velocity at points A and B as v_A and v_B we can write the equations for energy and momentum conservation as follows:

$$\frac{Mv_A^2}{2} - G\frac{MM_M}{R} = \frac{Mv_B^2}{2} - G\frac{MM_M}{R_M} \qquad (2)$$

$$Mv_AR = Mv_BR_M \qquad (3)$$

Solving equations (2) and (3) jointly we find

$$v_A = \sqrt{2G\frac{M_M R_M}{R(R+R_M)}}$$

Taking (1) into account, we get

$$v_A = v_0\sqrt{\frac{2R_M}{R+R_M}}.$$

Thus the rocket velocity change Δv at point A must be

$$\Delta v = v_0 - v_A = v_0\left(1 - \sqrt{\frac{2R_M}{R+R_M}}\right) = v_0\left(1 - \sqrt{\frac{2R_M}{2R_M + h}}\right) = 24 m/s.$$

Since the engine switches on for a short time the momentum conservation low in the system "rocket-fuel" can be written in the form

$$(M - m_1)\Delta v = m_1 u$$

where m_1 is the burnt fuel mass.

This yields

$$m_1 = \frac{\Delta v}{u + \Delta v}$$

Allow for $\Delta v << u$ we find

$$m_1 \approx \frac{\Delta v}{u} M = 29\text{kg}$$

2) In the second case the vector $\vec{v}_2$ is directed perpendicular to the vector $\vec{v}_0$ thus giving

$$\vec{v}_A = \vec{v}_0 + \Delta\vec{v}_2, \qquad v_A = \sqrt{v_0^2 + \Delta v_2^2} \quad .$$

Based on the energy conservation law in this case the equation can be written as

$$\frac{M\left(v_0^2 + \Delta v_2^2\right)}{2} - \frac{GMM_M}{R} = \frac{Mv_C^2}{2} - \frac{GMM_M}{R_M} \qquad (4)$$

and from the momentum conservation law

$$Mv_0 R = Mv_C R_M \ . \qquad (5)$$

Solving equations (4) and (5) jointly and taking into account (1) we find

$$\Delta v_2 = \sqrt{g_M \frac{\left(R - R_M\right)^2}{R}} = h\sqrt{\frac{g_M}{R_M + h}} \approx 97\text{m/s} \ .$$

Using the momentum conservation law we obtain

$$m_2 = \frac{\Delta v_2}{u} M \approx 116\text{kg} \ .$$