Solution of the Theoretical Problem 2

1) The voltage U_0 of the lamp of resistance R_0 is adjusted using the rheostat of resistance R. Using the Kirchhoff laws one obtains:

$$I = \frac{U_0}{R} + \frac{U_0}{R - R_x}, \qquad (1)$$

where $R - R_x$ is the resistance of the part of the rheostat, parallel connected to the lamp, R_x is the resistance of the rest part,

$$U_0 = E - IR_x \qquad (2)$$

The efficiency η of such a circuit is

$$\eta = \frac{P_{lamp}}{P_{accum.}} = \frac{U_0^2 / r}{IE} = \frac{U_0^2}{RIE}. \qquad (3)$$

From eq. (3) it is seen that the maximal current, flowing in the rheostat, is determined by the minimal value of the efficiency:

$$I_{\max} = \frac{U_0^2}{RE\eta_{\min}} = \frac{U_0^2}{RE\eta_0}. \qquad (4)$$

The dependence of the resistance of the rheostat R on the efficiency η can determined replacing the value for the current I , obtained by the eq. (3), $I = \frac{U_0^2}{RE\eta}$, in the eqs. (1) and (2):

$$\frac{U_0}{RE\eta} = \frac{1}{R_0} + \frac{1}{R - R_x}, \qquad (5)$$

$$R_x = (E - U_0)\frac{RE\eta}{U_0^2}. \qquad (6)$$

Then

$$R = R_0 \eta \frac{E^2}{U_0^2} \frac{1 + \eta(1 - \frac{E}{U_0})}{1 - \frac{E}{U_0}\eta}. \qquad (7)$$

To answer the questions, the dependence $R(\eta)$ must be investigated. By this reason we find the first derivative R'_η:

$$R'_\eta \propto \left(\frac{\eta + \eta^2 (1 - \frac{E}{U_0})}{1 - \frac{E}{U_0}\eta} \right)' \propto$$

$$\propto 1 + 2\eta(1 - \frac{E}{U_0})(1 - \frac{E}{U_0}\eta) + \left[\eta + \eta^2 (1 - \frac{E}{U_0}) \right] \frac{E}{U_0} = \eta(2 - \frac{E}{U_0}\eta)(1 - \frac{E}{U_0}) + 1.$$

$\eta < 1$, therefore the above obtained derivative is positive and the function $R(\eta)$ is increasing. It means that the efficiency will be minimal when the rheostat resistance is minimal. Then

$$R \geq R_{\min} = R_0 \eta_0 \frac{E^2}{U_0^2} \frac{1 + \eta_0 (1 - \frac{E}{U_0})}{1 - \frac{E}{U_0} \eta_0} \approx 8.53\Omega .$$

The maximal current I_{max} can be calculated using eq. (4). The result is: $I_{max} \approx 660$ mA.

2) As the function $R(\eta)$ is increasing one, $\eta \to \eta_{\max}$, when $R \to \infty$. In this case the total current I will be minimal and equal to $\frac{U_0}{R}$. Therefore the maximal efficiency is

$$\eta_{\max} = \frac{U_0}{E} = 0.75$$

This case can be realized connecting the rheostat in the circuit using only two of its three plugs. The used part of the rheostat is R_1:

$$R_1 = \frac{E - U_0}{I_0} = \frac{E - U_0}{U_0} R_0 \approx 0.67\Omega .$$