Problem 3

a) The necessary condition for the space-probe to escape from the Solar system is that the sum of its kinetic and potential energy in the Sun's gravitational field is larger than or equal to zero:

$$\tfrac{1}{2}mv_a^2 - \frac{GmM}{R_E} \geq 0\,,$$

where m is the mass of the probe, v_a its velocity relative to the Sun, M the mass of the Sun, R_E the distance of the Earth from the Sun and G the gravitational constant. Using the expression for the velocity of the Earth, $v_E = \sqrt{GM/R_E}$, we can eliminate G and M from the above condition:

$$v_a^2 \geq \frac{2GM}{R_E} = 2v_E^2\,. \qquad \text{(1p.)}$$

Let v_a' be the velocity of launching relative to the Earth and ϑ the angle between $\vec{v}_E$ and $\vec{v}_a'$ (Fig. 7). Then from $\vec{v}_a = \vec{v}_a' + \vec{v}_E$ and $v_a^2 = 2v_E^2$ it

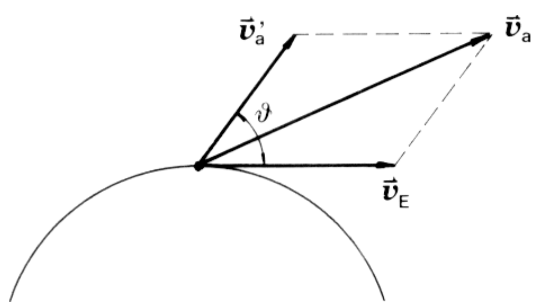

Fig. 7

follows:

$$v_a'^2 + 2v_a' v_E \cos\vartheta - v_E^2 = 0$$

and

$$v_a' = v_E\left[-\cos\vartheta + \sqrt{1+\cos^2\vartheta}\right].$$

The minimum velocity is obtained for $\vartheta = 0$:

$$v_a' = v_E(\sqrt{2}-1) = 12.3\ \text{km/s}\,. \qquad \text{(1p.)}$$

b) Let v_b' and v_b be the velocities of launching the probe in the Earth's and Sun's system of reference respectively. For the solution (a), $v_b = v_b' + v_E$. From the conservation of angular momentum of the probe:

$$mv_bR_E = mv_{\parallel}R_M \qquad \text{(1p.)}$$

and the conservation of energy:

$$\tfrac{1}{2}mv_b^2 - \frac{GmM}{R_E} = \tfrac{1}{2}m(v_\parallel^2 + v_\perp^2) - \frac{GmM}{R_M} \qquad (1\text{p.})$$

we get for the, parallel component of the velocity (Fig. 8):

$$v_\parallel = (v_b' + v_E)k\,,$$

and for the perpendicular component:

$$v_\perp = \sqrt{(v_b' + v_E)^2(1-k^2) - 2v_E^2(1-k)}\,. \qquad (1\text{p.})$$

where $k = R_E/R_M$.

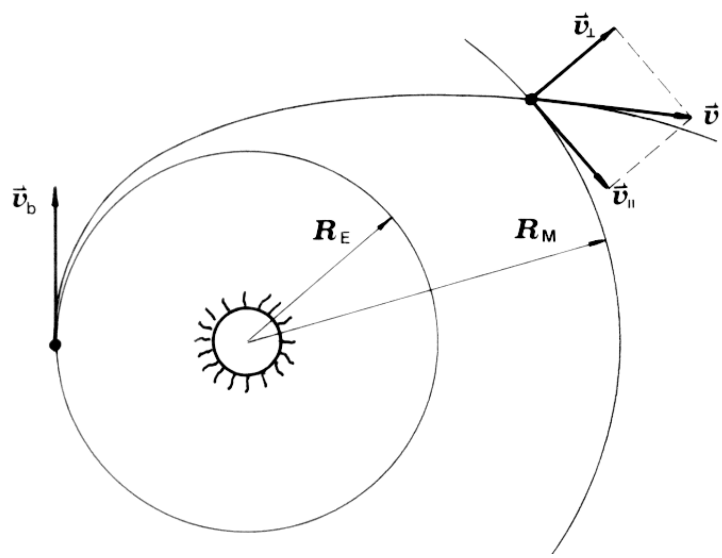

Fig. 8

c) The minimum velocity of the probe in the Mars' system of reference to escape from the Solar system, is $v_s'' = v_M(\sqrt{2}-1)$, in the direction parallel to the Mars orbit (v_M is the Mars velocity around the Sun). The role of Mars is therefore to change the velocity of the probe so that it leaves its gravitational field with this velocity.

(1 p.)

In the Mars' system, the energy of the probe is conserved. That is, however, not true in the Sun's system in which this encounter can be considered as an elastic collision between Mars and the probe. The velocity of the probe before it enters the gravitational field of Mars is therefore, in the Mars' system, equal to the velocity with which the probe leaves its gravitational field. The components of the former velocity are $v''_\perp = v_\perp$ and $v''_\parallel = v_\parallel - v_M$, hence:

$$v'' = \sqrt{{v''_\parallel}^2 + {v''_\perp}^2} = \sqrt{v_\perp^2 + (v_\parallel - v_M)^2} = v''_s\,. \qquad \text{(1p.)}$$

Using the expressions for $v_\perp$ and $v_\parallel$ from (b), we can now find the relation between the launching velocity from the Earth, v'_b, and the velocity v''_s, $v''_s = v_M(\sqrt{2}-1)$:

$$(v'_b+v_E)^2(1-k^2)-2v_E^2(1-k)+(v'_b+v_E)^2k^2-2v_M(v'_b+v_E)k = v_M^2(2-2\sqrt{2})\,.$$

The velocity of Mars round the Sun is $v_M = \sqrt{GM/R_M} = \sqrt{k}\,v_E$, and the equation for v'_b takes the form:

$$(v'_b + v_E)^2 - 2\sqrt{k}^3 v_E(v'_b + v_E) + (2\sqrt{2}\,k - 2)v_E^2 = 0\,. \qquad \text{(1p.)}$$

The physically relevant solution is:

$$v'_b = v_E\left[\sqrt{k}^3 - 1 + \sqrt{k^3 + 2 - 2\sqrt{2}\,k}\right] = 5.5\ \text{km/s}\,. \qquad \text{(1p.)}$$

d) The fractional saving of energy is:

$$\frac{W_a - W_b}{W_a} = \frac{{v'_a}^2 - {v'_b}^2}{{v'_a}^2} = 0.80\,,$$

where W_a and W_b are the energies of launching in scheme (i) and in scheme (ii), respectively. (1 p.)