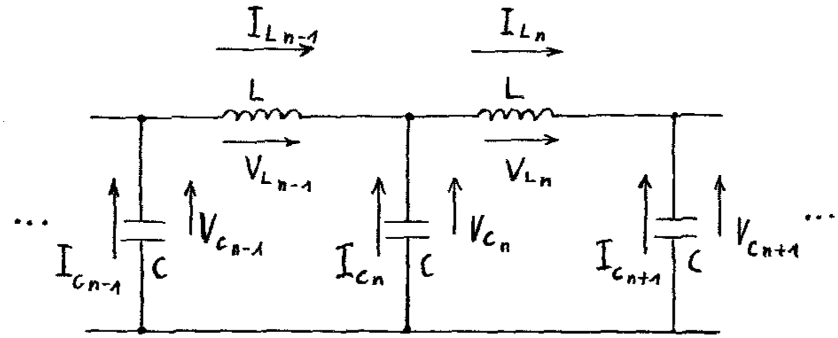
Solution of problem 3:

a)



Current law: $$I_{L_{n-1}} + I_{C_n} - I_{L_n} = 0 \quad (1)$$

Voltage law: $$V_{C_{n-1}} + V_{L_{n-1}} - V_{C_n} = 0 \quad (2)$$

Capacitive voltage drop: $V_{C_{n-1}} = \frac{1}{\omega \cdot C} \cdot \tilde{I}_{C_{n-1}}$ (3)

Note: In eq. (3) $\tilde{I}_{C_{n-1}}$ is used instead of $I_{C_{n-1}}$ because the current leads the voltage by 90°.

Inductive voltage drop: $V_{L_{n-1}} = \omega \cdot L \cdot \tilde{I}_{L_{n-1}}$ (4)

Note: In eq. (4) $\tilde{I}_{L_{n-1}}$ is used instead of $I_{L_{n-1}}$ because the current lags behind the voltage by 90°.

The voltage V_{C_n} is given by: $V_{C_n} = V_0 \cdot \sin\left(\omega \cdot t + n \cdot \varphi\right)$ (5)

Formula (5) follows from the problem.

From eq. (3) and eq. (5): $I_{C_n} = \omega \cdot C \cdot V_0 \cdot \cos\left(\omega \cdot t + n \cdot \varphi\right)$ (6)

From eq. (4) and eq. (2) and with eq. (5)

$$I_{L_{n-1}} = \frac{V_0}{\omega \cdot L} \cdot \left[2 \cdot \sin\left(\omega \cdot t + \left(n - \frac{1}{2} \right) \cdot \varphi \right) \cdot \sin\frac{\varphi}{2} \right] \quad (7)$$

$$I_{L_n} = \frac{V_0}{\omega \cdot L} \cdot \left[2 \cdot \sin\left(\omega \cdot t + \left(n + \frac{1}{2} \right) \cdot \varphi \right) \cdot \sin\frac{\varphi}{2} \right] \quad (8)$$

Eqs. (6), (7) and (8) must satisfy the current law. This gives the dependence of φ on ω, L and C.

$$0 = V_0 \cdot \omega \cdot C \cdot \cos\left(\omega \cdot t + n \cdot \varphi\right) + 2 \cdot \frac{V_0}{\omega \cdot L} \cdot \sin\frac{\varphi}{2} \cdot \left[2 \cdot \cos\left(\omega \cdot t + n \cdot \varphi\right) \cdot \sin\left(-\frac{\varphi}{2} \right) \right]$$

This condition must be true for any instant of time. Therefore it is possible to divide by $V_0 \cdot \cos(\omega \cdot t + n \cdot \varphi)$.

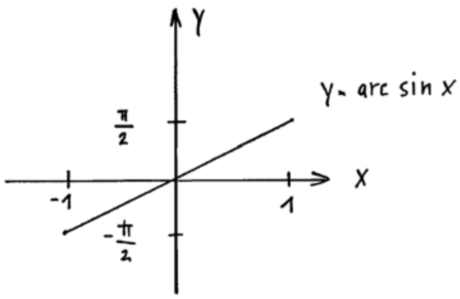
Hence $\omega^2 \cdot L \cdot C = 4 \cdot \sin^2\left(\frac{\varphi}{2}\right)$. The result is

$$\varphi = 2 \cdot \arcsin\left(\frac{\omega \cdot \sqrt{L \cdot C}}{2} \right) \quad \text{with} \quad 0 \le \omega \le \frac{2}{\sqrt{L \cdot C}} \quad (9).$$

b) The distance ℓ is covered in the time Δt thus the propagation velocity is

$$v = \frac{\ell}{\Delta t} = \frac{\omega \cdot \ell}{\varphi} \quad \text{or} \quad v = \frac{\omega \cdot \ell}{2 \cdot \arcsin\left(\frac{\omega \cdot \sqrt{L \cdot C}}{2} \right)} \quad (10)$$

c)



Slightly dependent means arc sin $\left(\frac{\omega\cdot\sqrt{L\cdot C}}{2}\right)\sim\omega$, since v is constant in that case.

This is true only for small values of ω. That means $\frac{\omega\cdot\sqrt{L\cdot C}}{2}\ll 1$ and therefore

$$v_0 = \frac{\ell}{\sqrt{L\cdot C}} \tag{11}$$

d) The energy is conserved since only inductances and capacitances are involved. Using the terms of a) one obtains the capacitive energy

$$W_C = \sum_n \frac{1}{2}\cdot C\cdot {V_{C_n}}^2 \tag{12}$$

and the inductive energy

$$W_L = \sum_n \frac{1}{2}\cdot L\cdot {I_{L_n}}^2 \tag{13}$$

From this follows the standard form of the law of conservation of energy

$$W_C = \sum_n \frac{1}{2}\left(C\cdot {V_{C_n}}^2 + L\cdot {I_{L_n}}^2\right) \tag{14}$$

The relation to mechanics is not recognizable in this way since two different physical quantities (V_{C_n} and I_{L_n}) are involved and there is nothing that corresponds to the relation between the locus x and the velocity v = $\dot{x}$.

To produce an analogy to mechanics the energy has to be described in terms of the charge Q, the current I = $\dot{Q}$ and the constants L and C. For this purpose the voltage V_{C_n} has to be expressed in terms of the charges Q_{L_n} passing through the coil.

One obtains:

$$W = \sum_n \left[\underbrace{\frac{L}{2} \cdot \dot{Q}_{L_n}^{\;2}}_{A} + \underbrace{\frac{1}{2 \cdot C} \left(Q_{L_n} - Q_{L_{n-1}} \right)^2}_{B} \right] \tag{15}$$

Mechanical analogue:

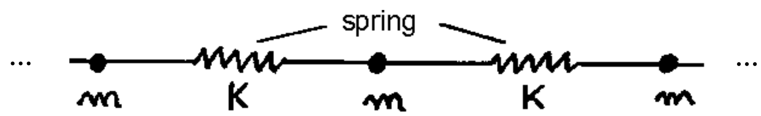
A (kinetic part): $\dot{Q}_{L_n} \longrightarrow v_n$; $L \longrightarrow m$

B (potential part): $Q_{L_n} \longrightarrow x_n$

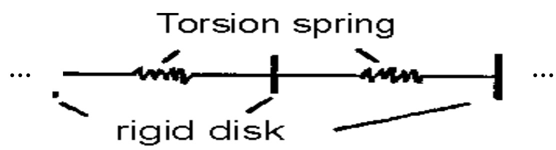
x_n: displacement and v_n: velocity.

However, Q_{L_n} could equally be another quantity (e.g. an angle). L could be e.g. a moment of inertia.

From the structure of the problems follows: Interaction only with the nearest neighbour (the force rises linearly with the distance). A possible model could be:



Another model is:



Experimental Problems

Problem 4: Refractive indices

Find the refractive indices of a prism, n_p, and a liquid, n_l. Ignore dispersion.

a) Determine the refractive index n_p of a single prism by two different experimental methods.
Illustrate your solution with accurate diagrams and deduce the relations necessary to calculate the refractive index. (One prism only should be used).