Solution to problem 3

PART A

At the beginning one should notice that the kinetic energy of the electron accelerated with the potential difference U = 511 kV equals to its rest energy E_0. Therefore, at least in the case of the electron, the laws of the classical physics cannot be applied. It is necessary to use relativistic laws.

The relativistic equation of motion of a particle with the charge e in the magnetic field $\mathbf{B}$ has the following form:

$$\frac{d}{dt}\mathbf{p} = \mathbf{F}_L$$

where $\mathbf{p} = m_0\gamma\mathbf{v}$ denotes the momentum of the particle (vector) and

$$\mathbf{F}_L = e\mathbf{v}\times\mathbf{B}$$

is the Lorentz force (its value is evB and its direction is determined with the right hand rule). m_0 denotes the (rest) mass of the particle and v denotes the velocity of the particle. The quantity γ is given by the formula:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

The Lorentz force $\mathbf{F}_L$ is perpendicular to the velocity $\mathbf{v}$ of the particle and to its momentum $\mathbf{p} = m_0\gamma\mathbf{v}$. Hence,

$$\mathbf{F}_L \cdot \mathbf{v} = \mathbf{F}_L \cdot \mathbf{p} = 0.$$

Multiplying the equation of motion by $\mathbf{p}$ and making use of the hint given in the text of the problem, we get:

$$\frac{1}{2}\frac{d}{dt}p^2 = 0.$$

It means that the value of the particle momentum (and the value of the velocity) is constant during the motion:

$$p = m_0 v\gamma = \text{ const;} \qquad v = \text{const.}$$

The same result can be obtained without any formulae in the following way:

The Lorentz force $\mathbf{F}_L$ is perpendicular to the velocity $\mathbf{v}$ (and to the momentum p as $\mathbf{p} = m_0\gamma\mathbf{v}$) and, as a consequence, to the trajectory of the particle. Therefore, there is no force that could change the component of the momentum tangent to the trajectory. Thus, this component, whose value is equal to the length of $\mathbf{p}$, should be constant: $\mathbf{p} = \text{const}$. (The same refers to the component of the velocity tangent to the trajectory as $\mathbf{p} = m_0\gamma\mathbf{v}$).

Let s denotes the path passed by the particle along the trajectory. From the definition of the velocity, we have:

$$\frac{ds}{dt} = v.$$

Using this formula, we can rewrite the equation of motion as follows:

$$v\frac{d}{ds}\mathbf{p} = \frac{ds}{dt}\frac{d}{ds}\mathbf{p} = \frac{d}{dt}\mathbf{p} = \mathbf{F}_L\,,$$

$$\frac{d}{ds}\mathbf{p} = \frac{\mathbf{F}_L}{v}\,.$$

Dividing this equation by p and making use of the fact that p = const, we obtain:

$$v\frac{d}{ds}\frac{\mathbf{p}}{p} = \frac{\mathbf{F}_L}{vp}$$

and hence

$$\frac{d}{ds}\mathbf{t} = \frac{\mathbf{F}_L}{vp}$$

where $\mathbf{t} = \mathbf{p}/p = \mathbf{v}/v$ is the versor (unit vector) tangent to the trajectory. The above equation is exactly the same for both electrons and protons if and only if the vector quantity:

$$\frac{\mathbf{F}_L}{vp}$$

is the same in both cases.

Denoting corresponding quantities for protons with the same symbols as for the electrons, but with primes, one gets that the condition, under which both electrons and protons can move along the same trajectory, is equivalent to the equality:

$$\frac{\mathbf{F}_L}{vp} = \frac{\mathbf{F}'_L}{v'p'}\,.$$

However, the Lorentz force is proportional to the value of the velocity of the particle, and the directions of any two vectors of the following three: **t** (or **v**), $\mathbf{F}_\mathrm{L}$, **B** determine the direction of the third of them (right hand rule). Therefore, the above condition can be written in the following form:

$$\frac{e\mathbf{B}}{p} = \frac{e'\mathbf{B}'}{p'}\,.$$

Hence,

$$\mathbf{B}' = \frac{e}{e'}\frac{p'}{p}\mathbf{B} = \frac{p'}{p}\mathbf{B}\,.$$

This means that at any point the direction of the field **B** should be conserved, its orientation should be changed into the opposite one, and the value of the field should be multiplied by the same factor p'/p. The magnetic field **B** is a vector sum of the magnetic fields of the coils that are arbitrarily distributed in the space. Therefore, each of this fields should be scaled with the same factor $-p'/p$. However, the magnetic field of any coil is proportional to the current flowing in it. This means that the required scaling of the fields can only be achieved by the scaling of all the currents with the same factor $-p'/p$:

$$i'_n = -\frac{p'}{p}i_n\,.$$

Now we shall determine the ratio p'/p. The kinetic energies of the particles in both cases are the same; they are equal to $E_k = e|U| = 511$ keV. The general relativistic relation between the total energy E of the particle with the rest energy E_0 and its momentum p has the following form:

$$E^2 = E_0^2 + p^2c^2$$

where c denotes the velocity of light.

The total energy of considered particles is equal to the sum of their rest and kinetic energies:

$$E = E_0 + E_k \,.$$

Using these formulae and knowing that in our case $E_k = e|U| = E_e$, we determine the momenta of the electrons (p) and the protons (p'). We get:

a) electrons:

$$(E_e + E_e)^2 = E_e^2 + p^2c^2,$$

$$p = \frac{E_e}{c}\sqrt{3}.$$

b) protons

$$(E_p + E_e)^2 = E_p^2 + p'^2 c^2,$$

$$p' = \frac{E_e}{c}\sqrt{\left(\frac{E_p}{E_e}+1\right)^2 - \left(\frac{E_p}{E_e}\right)^2}\,.$$

Hence,

$$\frac{p'}{p} = \frac{1}{\sqrt{3}}\sqrt{\left(\frac{E_p}{E}+1\right)^2 - \left(\frac{E_p}{E_e}\right)^2} \approx 35.0$$

and

$$i'_n = -35.0\, i_n \,.$$

It is worthwhile to notice that our protons are 'almost classical', because their kinetic energy $E_k (= E_e)$ is small compared to the proton rest energy E_p. Thus, one can expect that the momentum of the proton can be determined, with a good accuracy, from the classical considerations. We have:

$$E_e = E_k = \frac{p'^2}{2m_p} = \frac{p'^2 c^2}{2m_p c^2} = \frac{p'^2 c^2}{2E_p},$$

$$p' = \frac{1}{c}\sqrt{2E_e E_p}\,.$$

On the other hand, the momentum of the proton determined from the relativistic formulae can be written in a simpler form since E_p/E_e » 1. We get:

$$p' = \frac{E_e}{c}\sqrt{\left(\frac{E_p}{E_e}+1\right)^2-\left(\frac{E_p}{E_e}\right)^2} = \frac{E_e}{c}\sqrt{2\frac{E_p}{E_e}+1} \approx \frac{E_e}{c}\sqrt{2\frac{E_p}{E_e}} = \frac{1}{c}\sqrt{2E_eE_p}.$$

In accordance with our expectations, we have obtained the same result as above.

PART B

The resolving power of the microscope (in the meaning mentioned in the text of the problem) is proportional to the wavelength, in our case to the length of the de Broglie wave:

$$\lambda = \frac{h}{p}$$

where h denotes the Planck constant and p is the momentum of the particle. We see that λ is inversely proportional to the momentum of the particle. Therefore, after replacing the electron beam with the proton beam the resolving power will be changed by the factor $p/p' \approx 1/35$. It means that our proton microscope would allow observation of the objects about 35 times smaller than the electron microscope.

Marking scheme

1. the relativistic equation of motion	1 point
2. independence of p and v of the time	1 point
3. identity of $e\mathbf{B}/p$ in both cases	2 points
4. scaling of the fields and the currents with the same factor	1 point
5. determination of the momenta (relativistically)	1 point
6. the ratio of the momenta (numerically)	1 point
7. proportionality of the resolving power to λ	1 point
8. inverse proportionality of λ to p	1 point
9. scaling of the resolving power	1 point

Remarks and typical mistakes in the pupils' solutions

Some of the participants tried to solve the problem by using laws of classical mechanics only. Of course, this approach was entirely wrong. Some students tried to find the required condition by equating "accelerations" of particles in both cases. They understood the "acceleration" of the particle as a ratio of the force acting on the particle to the "relativistic" mass of the particle. This approach is incorrect. First, in relativistic physics the relationship between force and acceleration is more complicated. It deals with not one "relativistic" mass, but with two "relativistic" masses: transverse and longitudinal. Secondly, identity of trajectories need not require equality of accelerations.

The actual condition, i.e. the identity of $e\mathbf{B}/p$ in both cases, can be obtained from the following two requirements:

1° in any given point of the trajectory the curvature should be the same in both cases;

2° in the vicinity of any given point the plane containing a small arc of the trajectory should be oriented in space in both cases in the same way.

Most of the students followed the approach described just above. Unfortunately, many forgot about the second requirement (they neglected the vector character of the quantity $e\mathbf{B}/p$).