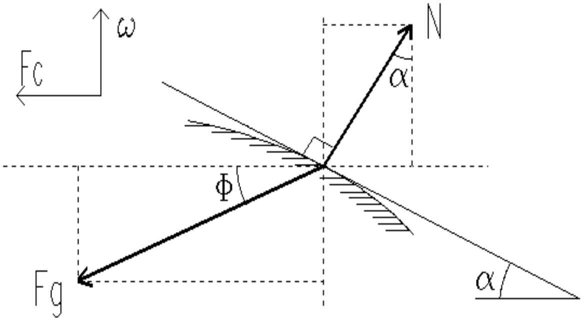
Solution of question 3.

a - *1st method*

For equilibrium we have $F_c = F_g + N$ where N is normal to the surface.



Resolving into horizontal and vertical components, we find:

$$\begin{matrix} F_g.\cos(\phi) = F_c + N.\sin(\alpha) \\ F_g.\sin(\phi) = N.\cos(\alpha) \end{matrix} \rightarrow F_g.\cos(\phi) = F_c + F_g.\sin(\phi).tg(\alpha)$$

From:

$$F_g = \frac{G.M}{r^2}\ ,\quad F_c = \omega^2.r\ ,\quad x = r.\cos(\phi)\ ,\quad y = r.\sin(\phi)\ \ en\ \ tg(\alpha) = \frac{dy}{dx}$$

we find:

$$y.dy + \left(1 - \frac{\omega^2.r^3}{G.M}\right).x.dx = 0$$

where:

$$\frac{\omega^2.r^3}{G.M} \approx 7.10^{-4}$$

This means that, although r depends on x and y, the change in the factor in front of xdx is so slight that we can take it to be constant. The solution of Eq. (1) is then an ellipse:

$$\frac{x^2}{r_e^2} + \frac{y^2}{r_p^2} = 1 \rightarrow \frac{r_p}{r_e} = \sqrt{1 - \frac{\omega^2.r^3}{G.M}} \approx 1 - \frac{\omega^2.r^3}{2.G.M}$$

and from this it follows that:

$$\epsilon = \frac{r_e - r_p}{r_e} = \frac{\omega^2.r^3}{2.G.M} \approx 3{,}7.10^{-4}$$

2nd method

For a point mass of 1 kg on the surface,

$$U_{pot} = -\frac{G.M}{r} \qquad U_{kin} = \frac{1}{2}.\omega^2.r^2.\cos^2(\phi)$$

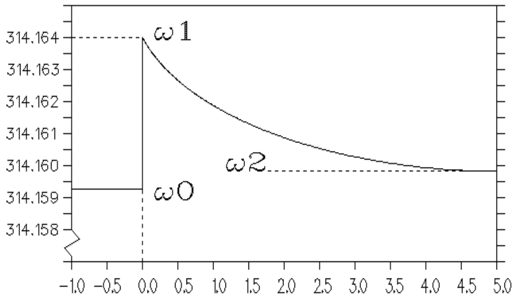
The form of the surface is such that $U_{pot} - U_{kin}$ = constant. For the equator ($\Phi = 0$, $r = r_e$) and for the pole ($\Phi = \pi/2$, $r = r_p$) we have:

$$\frac{G.M}{r_p} = \frac{G.M}{r_e} + \frac{1}{2}.\omega^2.r_e^2 \rightarrow \frac{r_e}{r_p} = 1 + \frac{\omega^2.r_e^3}{2.G.M}$$

Thus:

$$\epsilon = \frac{r_e - r_p}{r_e} = \frac{1 + \frac{\omega^2.r_e^3}{2.G.M} - 1}{1 + \frac{\omega^2.r_e^3}{2.G.M}} \approx \frac{\omega^2.r_e^3}{2.G.M} \approx 3{,}7.10^{-4}$$

b - As a consequence of the star-quake, the moment of inertia of the crust I_m decreases by ΔI_m.



From the conservation of angular momentum, we have:

$$I_m.\omega_0 = (I_m - \Delta I_m).\omega_1 \quad \rightarrow \quad \Delta I_m = I_m.\frac{\omega_1 - \omega_0}{\omega_1}$$

After the internal friction has equalized the angular velocities of the crust and the core, we have:

$$(I_m + I_c).\omega_0 = (I_m + I_c - \Delta I_m).\omega_2 \rightarrow \Delta I_m = (I_m + I_c).\frac{\omega_2 - \omega_0}{\omega_2}$$

$$\frac{I_m}{I_m + I_c} = \frac{(\omega_2 - \omega_0).\omega_1}{(\omega_1 - \omega_0).\omega_2} \rightarrow 1 - \frac{I_c}{I_m + I_c} = \frac{(\omega_2 - \omega_0).\omega_1}{(\omega_1 - \omega_0).\omega_2}$$

$$I \ (:)\ R^2$$

$$\rightarrow \frac{I_c}{I_m + I_c} = \frac{r_c^2}{r^2} \rightarrow \frac{r_c}{r} = \sqrt{1 - \frac{(\omega_2 - \omega_0).\omega_1}{(\omega_1 - \omega_0).\omega_2}} \approx 0.95$$

Marking breakdown

a	1st method	- expressions for the forces	:1
		- equation for the surface	:2
		- equation of ellipse	:1
		- flattening factor	:1
	2nd method	- energy equation	:4
		- flattening factor	:1
b	- conservation of angular momentum for crust		:1.5
	- conservation of angular momentum for crust and core		:1.5
	- moment of inertia for a sphere		:1
	- ratio r_c/r		:1