Solution Problem 1

a) *Calculation of the velocity at the instant before impact*

Equating the potential gravitational energy to the kinetic energy at the instant before impact we can arrive at the pre-impact velocity v_0:

$$mgh = \frac{mv_0^2}{2} \tag{1}$$

from which we may solve for v_0 as follows:

$$V_0 = \sqrt{2gh} \tag{2}$$

b) *Calculation of the vertical component of the velocity at the instant after impact*

Let v_{2x} and v_{2y} be the horizontal and vertical components, respectively, of the velocity of the mass center an instant after impact. The height attained in the vertical direction will be αh and then:

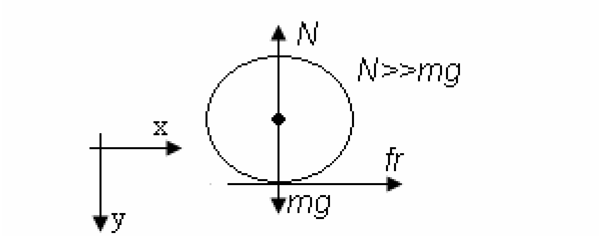
$$v_{2y}^2 = 2g\alpha h \tag{3}$$

from which, in terms of α (or the restitution coefficient $c = \sqrt{\alpha}$):

$$v_{2y} = \sqrt{2g\alpha h} = cv_0 \tag{4}$$

c) *General equations for the variations of linear and angular momenta in the time interval of the Impact*

Figure 1.2 shows the free body of the ball during impact



Considering that the linear impulse of the forces is equal to the variation of the linear momentum and that the angular impulse of the torques is equal to the variation of the angular momentum, we have:

$$I_y = \int_{t_1}^{t_2} N(t)dt = mv_0 + mv_{2y} = m(1+c)\sqrt{2gh} \tag{5}$$

$$I_x = \int_{t_1}^{t_2} f_r(t)dt = mv_{2x} \tag{6}$$

$$I_\theta = \int_{t_1}^{t_2} R f_r(t)dt = R\int_{t_1}^{t_2} f_r(t)dt = \mathrm{I}\,(\omega_o - \omega_2) \qquad (7)$$

Where I_x, I_y and I_θ are the linear and angular impulses of the acting forces and ω_2 is the angular velocity after impact. The times t_1 and t_2 correspond to the beginning and end of impact.

Variants

At the beginning of the impact the ball will always be sliding because it has a certain angular velocity ω_0. There are, then, two possibilities:

I. The entire impact takes place without the friction being able to spin the ball enough for it to stop at the contact point and go into pure rolling motion.

II. For a certain time $t \in (t_1, t_2)$, the point that comes into contact with the floor has a velocity equal to zero and from that moment the friction is zero. Let us look at each case independently.

Case I

In this variant, during the entire moment of impact, the ball is sliding and the friction relates to the normal force as:

$$f_r = \mu_k N(t) \qquad (8)$$

Substituting (8) in relations (6) and (7), and using (5), we find that:

$$\mathrm{I}_x = \mu_k \int_{t_1}^{t_2} N(t)dt = \mu_k \mathrm{I}_y = \mu_k (1+c)\sqrt{2gh} = mv_{2x} \qquad (9)$$

and:

$$\mathrm{I}_\theta = \mathrm{R}\,\mu_k \int_{t_1}^{t_1} N(t)dt = \mathrm{R}\,\mu_k\, m(1+c)\sqrt{2gh} = \mathrm{I}(\mu_0 - \mu_2) \qquad (10)$$

which can give us the horizontal component of the velocity v_{2x} and the final angular velocity in the form:

$$V_{2x} = \mu_k (1+c)\sqrt{2gh} \qquad (11)$$

$$\omega_2 = \omega_0 - \frac{\mu_k mR(1+c)}{I}\sqrt{2gh} \qquad (12)$$

With this we have all the basic magnitudes in terms of data. The range of validity of the solution under consideration may be obtained from (11) and (12). This solution will be valid whenever at the end of the impact the contact point has a velocity in the direction of the negative x. That is, if:

$$\omega_2 \mathrm{R} > v_{2x}$$

$$\omega_0 - \frac{\mu_k mR(1+c)}{I}\sqrt{2gh} > \frac{\mu_k (1+c)}{R}\sqrt{2gh}$$

$$\omega_0 > \frac{\mu_k \sqrt{2gh}}{R}(1+c)\left(\frac{mR^2}{I}+1\right) \qquad (13)$$

so, for angular velocities below this value, the solution is not valid.

Case II

In this case, rolling is attained for a time t between the initial time t_1 and the final time t_2 of the impact. Then the following relationship should exist between the horizontal component of the velocity v_{2x} and the final angular velocity:

$$\omega_2 \mathrm{R} = v_{2x} \qquad (14)$$

Substituting (14) and (6) in (7), we get that:

$$mRv_{2x} = I\left(\omega_0 - \frac{v_{2x}}{R}\right) \qquad (15)$$

which can be solved for the final values:

$$V_{2x} = \frac{I\omega_0}{mR + \frac{I}{R}} = \frac{I\omega_0 R}{mR^2 + I} = \frac{2}{7}\omega_0 R \qquad (16)$$

and:

$$\omega_2 = \frac{I\omega_0}{mR^2 + I} = \frac{2}{7}\omega_0 \qquad (17)$$

Calculation of the tangents of the angles

Case I

For tan θ we have, from (4) and (11), that:

$$\tan\theta = \frac{v_{2x}}{v_{2y}} = \frac{\mu_k(1+c)\sqrt{2gh}}{c\sqrt{2gh}} = \mu_k\frac{(1+c)}{c}$$

$$\tan\theta = \mu_k\frac{(1+c)}{c} \qquad (18)$$

i.e., the angle is independent of ω_0.

Case II

Here (4) and (16) determine for $\tan\theta$ that:

$$\tan\theta = \frac{v_{2x}}{v_{2y}} = \frac{I\omega_0 R}{I + mR^2}\ \frac{1}{c\sqrt{2gh}} = \frac{I\omega_0 R}{(I + mR^2)c\sqrt{2gh}}$$

$$\tan\theta = \frac{2\omega_0 R}{7c\sqrt{2gh}} \qquad (19)$$

then (18) and (19) give the solution (fig. 1.3).

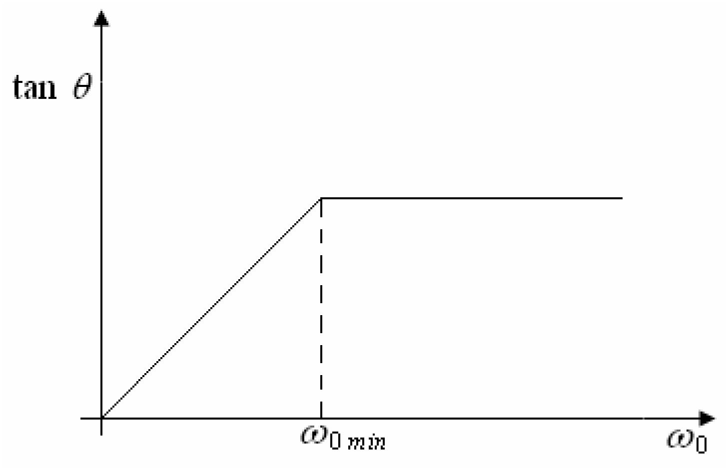


Figure 1.3

We see that θ does not depend on ω_o if $\omega_0 > \omega_{0\,min}$; where $\omega_{0\,min}$ is given as:

$$\omega_{o\,min} = \frac{\mu_k(1+c)\sqrt{2gh}\left(1 + \frac{mR^2}{I}\right)}{R}$$

$$\omega_{o\,min} = \frac{7\mu_k(1+c)\sqrt{2gh}}{2R} \tag{20}$$

Calculation of the distance to the second point of impact

Case I

The rising and falling time of the ball is:

$$t_v = 2\frac{v_{2y}}{g} = \frac{2c\sqrt{2gh}}{g} = 2c\sqrt{\frac{2h}{g}} \tag{21}$$

The distance to be found, then, is;

$$d_I = v_{2x}t_v = \mu_k(1+c)\sqrt{2gh}\,2c\sqrt{\frac{2h}{g}}$$

$$d_1 = 4\mu_k(1+c)\,ch \tag{22}$$

which is independent of ω_0.

Case II

In this case, the rising and falling time of the ball will be the one given in (21). Thus the distance we are trying to find may be calculated by multiplying t_v by the velocity v_{2x} so that:

$$d_{II} = v_{2x}t_v = \frac{I\omega_0}{mR^2+I}2c\sqrt{\frac{2h}{g}} = \frac{2\omega_0 Rc}{1+\frac{5}{2}}\sqrt{\frac{2h}{g}}$$

$$d_{II} = \frac{4}{7}c\sqrt{\frac{2h}{g}}R\omega_0$$

Thus, the distance to the second point of impact of the ball increases linearly with ω_0.

Marking Code

The point value of each of the sections is:

1.a	2 points
1.b	1.5 points
1.c	2 points
2.a	2 points
2.b	1.5 points
3	1 point