Solution Problem 2

Question a:

Let's call S the lab (observer) frame of reference associated with the observer that sees the loop moving with velocity v; S' to the loop frame of reference (the x' axis of this system will be taken in the same direction as $\vec{v}$; y' in the direction of side DA and z' axis, perpendicular to the plane of the loop). The axes of S are parallel to those of S' and the origins of both systems coincide at $t = 0$.

1. Side AB

S''_{AB} will be a reference frame where the moving balls of side AB are at rest. Its axes are parallel to those of S and S'. S'' has a velocity u with respect to S'.

According to the Lorentz contraction, the distance a, between adjacent balls of AB, measured in S'', is:

$$a_r = \frac{a}{\sqrt{1-\frac{u^2}{c^2}}} \tag{1}$$

(This result is valid for the distance between two adjacent balls that are in one of any sides, if a_r is measured in the frame of reference in which they are at rest).

Due to the relativistic sum of velocities, an observer in S sees the balls moving in AB with velocity:

$$u_{AB} = \frac{v+u}{1+\frac{uv}{c^2}} \tag{2}$$

So, because of Lorentz contraction, this observer will see the following distance between balls:

$$a_{AB} = \sqrt{1-\frac{u_{AB}^2}{c^2}}a_r \tag{3}$$

Substituting (1) and (2) in (3)

$$a_{AB} = \sqrt{\frac{1-\frac{v^2}{t^2}}{1+\frac{uv}{c^2}}}a \tag{4}$$

2. Side CD

For the observer in S, the speed of balls in CD is:

$$u_{CD} = \frac{v-u}{1-\frac{uv}{c^2}} \tag{5}$$

From the Lorentz contraction:

$$a_{CD} = \sqrt{1-\frac{u_{CD}^2}{c^2}}a_r \tag{6}$$

Substituting (1) and (5) in (6) we obtain:

$$a_{CD} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{uv}{c^2}}a \tag{7}$$

3. Side DA

In system S', at time $t_o^{'}$, let a ball be at $x_1^{'} = y_1^{'} = z_1^{'} = 0$. At the same time the nearest neighbour to this ball will be in the position $x_2^{'} = 0, y_2^{'} = a$, $z_2^{'} = 0.$

The space-time coordinates of this balls, referred to system S, are given by the Lorentz transformation:

$$x = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x'+vt')$$

y=y'

z=z' (8)

$$t = \frac{1}{\sqrt{1+\frac{v^2}{c^2}}}\left(t'+\frac{x'v'}{c^2}\right)$$

Accordingly, we have for the first ball in S:

$$x_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} vt'_o;\ y_1=0;\ z_1=0;\ t_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} t'_0 \tag{9}$$

(9)

And for the second:

$$x_2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} vt'_o; y_2 = a; z_2 = 0; t_2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} t'_o \tag{10}$$

As $t_1 = t_2$, the distance between two balls in S will be given by:

$a_{DA}=(x_2-x_1)^2 + (y_2 - y_1)^2 + (z_2-z_1)^2$ (11)

So:

$a_{AD}=a$ (12)

4. Side BC

If we repeat the same procedure as above, we can obtain that:

$a_{BC}=a$

(13)

Question b:

The charge of the wire forming any of the sides, in the frame of reference associated with the loop can be calculated as:

$$Q_{wire} = -\frac{L}{a}q \tag{14}$$

Because L/a is the number of balls in that side. Due to the fact that the charge in invariant, the same charge can be measured in each side of the wire in the lab (observer) frame of reference.

1. Side AB

The charge corresponding to balls in side AB is, in the lab frame of reference:

$$Q_{AB,\,balls} = \frac{L\sqrt{1-\frac{v^2}{c^2}}}{a_{AB}} - q \tag{15}$$

This is obtained from the multiplication of the number of balls in that side multiplied by the (invariant) charge of one ball. The numerator of the first factor in the right side of equation (15) is the contracted distance measured by the observer and the denominator is the spacing between balls in that side.

Replacing in (15) equation (4), we obtain:

$$Q_{AB,\,balls} = \left(\frac{1+uv}{c^2}\right)\frac{Lq}{a} \tag{16}$$

Adding (14) and (16) we obtain for the total charge of this side:

$$Q_{AB} = \frac{uv}{c^2}\frac{L}{a}q \tag{17}$$

2. Side CD

Following the same procedure we have that:

$$Q_{CD,\,balls} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{a_{CD}} - q = \left(1-\frac{uv}{c^2}\right)\frac{Lq}{a} \tag{18}$$

And adding (14) and (18) we obtain:

$$Q_{CD} = -\frac{uv}{c^2}\frac{L}{a}q \tag{19}$$

The length of these sides measured by the observer in S is L and the distance between balls is a, so:

$$Q_{BC,\,balls} = Q_{DA,\,balls} = \frac{Lq}{a} \tag{20}$$

Adding (14) and (20) we obtain:

$Q_{BC} = 0$ (21.1)
$Q_{DA}=0$ (21.2)
Question c:
There is electric force acting into the side AB equal to:

$$\vec{F}_{AB} = Q_{AB}\vec{E} = \left(\frac{uv}{c^2}\right)\frac{L}{a}q\vec{E} \tag{22}$$

and the electric force acting into the side CD is:

$$\vec{F}_{CD} = Q_{CD}\vec{E} = -\left(\frac{uv}{c^2}\right)\frac{L}{a}q\vec{E} \tag{23}$$

F_{CD} and F, form a force pair. So, from the expression for the torque for a force pair we have that (fig. 2.2):

$$M = \left|\vec{F}_{AB}\right| L\sin\theta \tag{24}$$

And finally:

$$M = \frac{uv}{c^2}\frac{L^2}{a}|q|\left|\vec{E}\right|\sin\theta \tag{25}$$

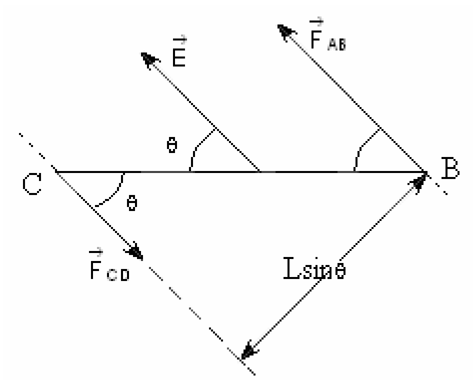

Fig 2.2

Question d:
Let's call V_{AB} and vCD the electrostatic in the points of sides AB and CD respectively. Then:
$W=V_{AB}Q_{AB} + V_{CD}Q_{CD}$ (26)

Let's fix cero potential (V=0) in a plane perpendicular to $\vec{E}$ and in an arbitrary distance R from side AB (fig. 2.3).

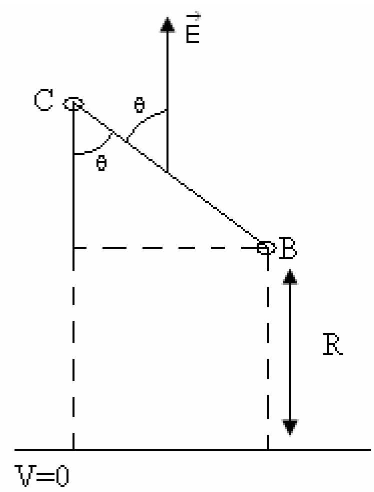

Figure 2.3

Then:
$W=-ERQ_{AB} – E(R+L\cos\theta)Q_{CD}$ (27)
But $Q_{CD}=-Q_{AB}$, so:
$W=-ELQ_{AB}\cos\theta$ (28)

Substituting (17) in (28) we obtain:

$$W = \frac{uvL^2 qE}{c^2 a}\cos\theta \qquad (29)$$

Marking Code

Grading for questions will be as follows:

a)4,5 points.

b)2,0 points.

c)1,5 points.

d)2,0 points.

These points are distributed in questions in the following way:

Question a:

1. Obtaining expressions (4) and (7) correctly: 3,0 points.
   Only one of them correct: 2,0 points.
2. Obtaining expressions (12) and (13) correctly including the necessary calculations to arrive to this results: 1,5 points.
   Only one of them correct: 1,0 points.

If the necessary calculations are not present: 0,8 point for both (12) and (13) correct; 0,5 points for only one of them correct.

Question b:

1. Obtaining expressions (17) and (19) correctly: 1,0 point.
   Only one of them correct: 1,0 point.
2. Obtaining expressions (21.1) and (21.2) correctly: 0,5 point.
   Only one them correct: 0,5 point.

Question d:

1. Obtaining expression (29) correctly: 2,0 points.

When the modulus of a vector is not present where necessary, the student will loose 0,2 points. When the modulus of q is not present where necessary the student will loose 0,1 points.