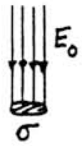
## Theoretical Problem 1 -- Solution

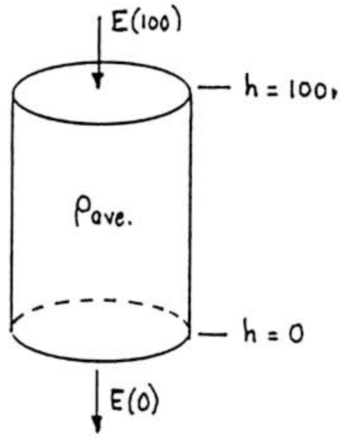
1) By Gauss' law, $\sigma = \varepsilon_0 E_0$.




$$\therefore \ \sigma = -8.85\ 10^{-12} \times 150$$
$$\approx -1.3 \times 10^{-9}\ \mathrm{C/m^2}.$$

$$Q = 4\pi R^2 \sigma = 4\pi \times (6.4 \cdot 10^6)^2 \times 1.3 \cdot 10^{-9} = -6.7 \cdot 10^5\ \mathrm{C}.$$

2) Consider a cylinder of cross-section $A$ with faces at heights of 0 and 100 m.




By Gauss' law, $E(0)\,A - E(100)\,A = q_{enclosed}/\varepsilon_0$

$$= \rho_{ave.} \times (100A)/\varepsilon_0.$$

$$\therefore \quad \rho_{ave.} = \frac{\varepsilon_0[E(0) - E(100)]}{100}$$

$$= \frac{8.85\ 10^{-12} \times 50}{100} \approx 4.4 \times 10^{-12}\ \mathrm{C/m^3}.$$

3) If a conductor contains $n$ charges per unit volume, each with charge $q$ and travelling with speed $v$, the current per unit area ($j$) is given by:

$$j = nqv.$$

Here, we have both positive and negative charges ($\pm e$). Clearly, with a downward electric field, the positive charges move downward and the negative charges move upward.
In the situation as described, only the positive charges can contribute to neutralization of the Earth's surface charge. Hence we have (taking downward as the positive direction for this purpose):

$$j = n_+ e\, v$$
$$\approx (6 \cdot 10^8) \times (1.6 \cdot 10^{-19}) \times (1.5 \cdot 10^{-4}\, E)$$
$$= 1.44 \times 10^{-14}\, E.$$

Now $j$ is the rate of change ($d\sigma/dt$) of the surface charge density $\sigma$, and $E$ (if defined as positive downward) is equal to $-\sigma/\varepsilon_0$. Thus the above equation can be written:

$$\frac{d\sigma}{dt} = -1.44 \cdot 10^{-14} \frac{\sigma}{\varepsilon_0} = -\frac{1.44 \cdot 10^{-14}}{8.85 \cdot 10^{-12}} \sigma = -1.63 \cdot 10^{-3} \sigma \approx -\frac{1}{600} \sigma.$$

This is just like the equation of radioactive decay. Its solution is an exponential decrease of $\sigma$ with time:

$$\sigma(t) = \sigma_0\, e^{-t/\tau}, \qquad \text{with } \tau \approx 600 \text{ sec.}$$

Putting $\sigma(t) = \sigma_0/2$ then gives $t = \tau \ln 2 = 0.693 \times 600 \approx 415$ sec $\approx 7$ min.

[A simpler approximate solution is to assume that $j$ remains constant at its initial value $j_0$:

$$j_0 = 1.44{\cdot}10^{-14}\,E_0 = 1.44{\cdot}10^{-14} \times 150 \approx 2.15 \times 10^{-12}\ \text{A/m}^2.$$

With $|\sigma_0| = 1.3\ 10^{-9}$ C/m$^2$ from part 1, we would then put:

$$|\sigma_0/2| = j_0 \times t,\ \text{giving}\ t = (0.65{\cdot}10^{-9})/(2.15{\cdot}10^{-12}) \approx 300\ \text{s} = 5\ \text{min.}]$$

4) If $t = 0$ is an instant at which the insulated quadrants are completely shielded, we have the following relations:

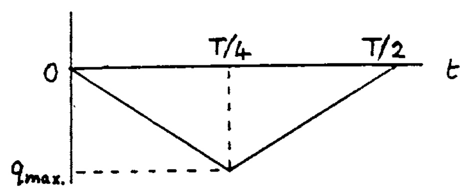
$$\text{For } 0 \le t \le \frac{T}{4},\ q(t) = -2\pi(r_2^2 - r_1^2)\varepsilon_0 E_0 \frac{t}{T}.$$

$$\text{For } \frac{T}{4} < t \le \frac{T}{2},\ q(t) = -\pi(r_2^2 - r_1^2)\varepsilon_0 E_0\left(1 - \frac{2t}{T}\right).$$

Corresponding variations occur during all the succeeding pairs of quarter-cycles.

The maximum (negative) induced charge is given by:

$$q_{max.} = -\frac{\pi}{2}(r_2^2 - r_1^2)\varepsilon_0 E_0.$$

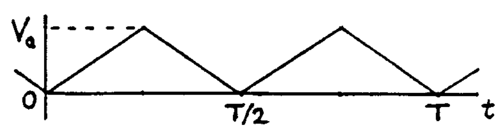



5) This question can be discussed without making a full circuit analysis. One only needs to realize that the rate of flow of charge into the amplifier is divided into a rate of charging of the capacitor, $C\,dV/dt$, and a conduction current, $V/R$, through the resistor. There are then two extreme situations, depending on whether the amount of charge lost by leakage during one quarter-period is small or large compared to $CV$.

(a) If $CV \gg (V/R) \times (T/4)$ -- i.e., $T = T_a \ll CR$ -- very little of the charge is carried away through $R$ during the time $T/4$. Thus, when the insulated quadrants are charged negatively through induction, an almost equal *positive* charge is given to $C$. Thus $V(t)$ rises almost linearly with $t$ between $t = 0$ and $t = T/4$, and then decreases almost linearly by an equal amount between $t = T/4$ and $t = T/2$. In this case,

$$V_{max.} = V_a \approx \frac{|q_{max.}|}{C},$$

where $q_{max.}$ has the value obtained in part 4.*




(b) If, however, $T = T_b \gg CR$ -- i.e., $CR \ll T_b$ -- most of the charge is quickly carried away through $R$. A constant positive current flows through $R$ when the magnitude of $q$ is increasing, and an equal negative current when the magnitude of $q$ is decreasing. The size

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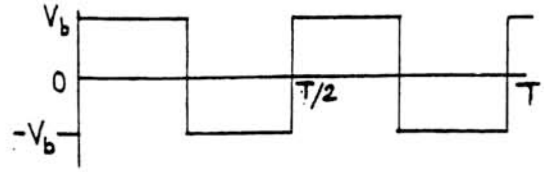
*Note: Ultimately (unless $CR$ is infinite) the form of $V_a$ will become a sawtooth varying symmetrically between $\pm\, q_{max.}/2C$. The statement of the problem avoids this complication by specifying that $V$ is measured *just after* the rotation has begun.

of this current is approximately equal to $|q_{max.}|/(T_b/4)$. The resulting voltage across $R$ is approximately constant during each quarter-period, and is alternately positive and negative. In this case,

$$V_{max.} = V_b \approx \frac{4\, q_{max.} R}{T_b}.$$




Putting these results together, we see that:

$$\frac{V_a}{V_b} \approx \frac{T_b}{4CR}.$$

6) We have $CR = 10^{-8} \times 2\ 10^{7} = 0.2$ s, and $T = 1/50 = 0.02$ s.
Thus $CR = 10 \times T$, which satisfies the criterion $CR \gg T$.
Therefore we can use the solution 5(a) above.
We have $A_{max.} = \frac{\pi}{2}(7^2 - 1^2) = 75\ \text{cm}^2 = 7.5 \times 10^{-3}\ \text{m}^2$.
$E_o = 150$ V/m $\rightarrow \sigma = \varepsilon_o E_o \approx 1.33 \times 10^{-9}$ C/m$^2$ (as in part 1).
$\therefore\ q_{max.} = 1.33 \cdot 10^{-9} \times 7.5\ 10^{-3} \approx 1.0 \times 10^{-11}$ C,

and so $V_{max.} = \frac{q_{max.}}{C} = \frac{1.0 \times 10^{-11}}{1.0 \times 10^{-8}} = 10^{-3}\ \text{V} = 1\ \text{mV}.$

Theoretical Problem 1: Grading Scheme

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|---|---|---|
| Part 1. | 1 point | (1/2 point for $\sigma_o$, 1/2 point for $Q$) |
| Part 2. | 1 point | |
| Part 3. | 2 points | (1/2 point for recognizing $j = nev$;<br>1/2 point for recognizing $j = d\sigma/dt$;<br>1/2 point for getting $\sigma(t) = \sigma_o\, e^{-t/\tau}$;<br>1/2 point for final numerical answer.)<br>[1 point maximum for using $t = \sigma_o/2j_o$.] |
| Part 4. | 1-1/2 points | (1/2 point for each equation;<br>1/2 point for graph.) |
| Part 5. | 3-1/2 points | (1 point for correct graphical form of (a);<br>1 point for correct graphical form of (b);<br>1-1/2 points for correct evaluation of $V_d/V_b$.) |
| Part 6. | 1 point | (1/2 point for recognizing that $T \ll CR$;<br>1/2 point for final answer) |