Theoretical Question 1

Gravitational Red Shift and the Measurement of Stellar Mass

(a) (3 marks)
A photon of frequency f possesses an effective inertial mass m determined by its energy. Assume that it has a gravitational mass equal to this inertial mass. Accordingly, a photon emitted at the surface of a star will lose energy when it escapes from the star's gravitational field. Show that the frequency shift Δf of the photon when it escapes from the surface of the star to infinity is given by

$$\frac{\Delta f}{f} \simeq -\frac{GM}{Rc^2}$$

for $\Delta f \ll f$ where:

- $G =$ gravitational constant
- $R =$ radius of the star
- $c =$ velocity of light
- $M =$ mass of the star.

Thus, the red-shift of a known spectral line measured a long way from the star can be used to measure the ratio M/R. Knowledge of R will allow the mass of the star to be determined.

(b) (12 marks)
An unmanned spacecraft is launched in an experiment to measure both the mass M and radius R of a star in our galaxy. Photons are emitted from He^+ ions on the surface of the star. These photons can be monitored through resonant absorption by He^+ ions contained in a test chamber in the spacecraft. Resonant absorption accors only if the He^+ ions are given a velocity towards the star to allow exactly for the red shifts.

As the spacecraft approaches the star radially, the velocity relative to the star ($v = \beta c$) of the He^+ ions in the test chamber at absorption resonance is measured as a function of the distance d from the (nearest) surface of the star. The experimental data are displayed in the accompanying table.

Fully utilize the data to determine graphically the mass M and radius R of the star. There is no need to estimate the uncertainties in your answer.

Data for Resonance Condition

Velocity parameter	$\beta = v/c$ $(\times 10^{-5})$	3.352	3.279	3.195	3.077	2.955
Distance from surface of star	d $(\times 10^{8}\text{m})$	38.90	19.98	13.32	8.99	6.67

(c) (5 marks)
In order to determine R and M in such an experiment, it is usual to consider the frequency correction due to the recoil of the emitting atom. [Thermal motion causes emission lines to be broadened without displacing emission maxima, and we may therefore assume that all thermal effects have been taken into account.]

(i) (4 marks)
Assume that the atom decays at rest, producing a photon and a recoiling atom. Obtain the relativistic expression for the energy hf of a photon emitted in terms of ΔE (the difference in rest energy between the two atomic levels) and the initial rest mass m_0 of the atom.

(ii) (1 mark)
Hence make a numerical estimate of the relativistic frequency shift $\left(\frac{\Delta f}{f}\right)_{\text{recoil}}$ for the case of He^+ ions.

Your answer should turn out to be much smaller than the gravitational red shift obtained in part (*b*).

Data:

Velocity of light	c	$=$	$3.0 \times 10^8 \text{ms}^{-1}$
Rest energy of He	m_0c^2	$=$	$4 \times 938(\text{MeV})$
Bohr energy	E_n	$=$	$-\dfrac{13.6Z^2}{n^2}(\text{eV})$
Gravitational constant	G	$=$	$6.7 \times 10^{-11} \text{Nm}^2\text{kg}^{-2}$