Solution Problem 2

a) The potential energy gain eV is converted into kinetic energy. Thus

$$\tfrac{1}{2} mv^2 = eV \qquad \text{(non-relativistically)}$$

$$\frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = eV \qquad \text{(relativistically).}$$

Hence

$$v = \begin{cases} \sqrt{2eV/m} & \text{(non - relativistically)} \\ c\sqrt{1 - \left(\dfrac{mc^2}{mc^2 + eV}\right)^2} & \text{(relativistically).} \end{cases} \qquad (1)$$

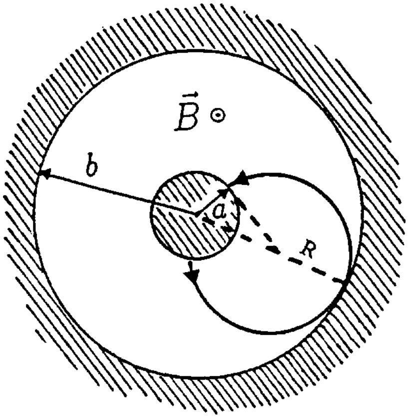
b) When $V = 0$ the electron moves in a homogeneous static magnetic field. The magnetic Lorentz force acts orthogonal to the velocity and the electron will move in a circle. The initial velocity is tangential to the circle.

The radius R of the orbit (the "cyclotron radius") is determined by equating the centripetal force and the Lorentz force:

i.e.

$$eBv_0 = \frac{mv_0^2}{R} \ ,$$

$$B = \frac{mv_0}{eR} \ . \qquad (2)$$



From the figure we see that in the critical case the radius R of the circle satisfies

$$\sqrt{a^2 + R^2} = b - R$$

By squaring we obtain

i.e.

$$a^2 + R^2 = b^2 - 2bR + R^2,$$

$$R = (b^2 - a^2)/2b \ .$$

Insertion of this value for the radius into the expression (2) gives the critical field

$$B_c = \frac{mv_0}{eR} = \frac{2bmv_0}{(b^2 - a^2)e}.$$

c) The change in angular momentum with time is produced by a torque. Here the azimuthal component F_ϕ of the Lorentz force $\vec{F} = (-e)\vec{B}\times\vec{v}$ provides a torque $F_\phi r$. It is only the radial component $v_r = dr/dt$ of the velocity that provides an azimuthal Lorentz force. Hence

$$\frac{dL}{dt} = eBr\frac{dr}{dt},$$

which can be rewritten as

$$\frac{d}{dt}(L - \frac{eBr^2}{2}) = 0.$$

Hence

$$C = \underline{\underline{L - \tfrac{1}{2} eBr^2}} \tag{3}$$

is constant during the motion. The dimensionless number k in the problem text is thus $k = \underline{1/2}$.

d) We evaluate the constant C, equation (3), at the surface of the inner cylinder and at the maximal distance r_{m} :

$$0 - \tfrac{1}{2} eBa^2 = mvr_m - \tfrac{1}{2} eBr_m^2$$

which gives

$$v = \underline{\frac{eB(r_m^2 - a^2)}{2mr_m}}. \tag{4}$$

Alternative solution: One may first determine the electric potential $V(r)$ as function of the radial distance. In cylindrical geometry the field falls off inversely proportional to r, which requires a logarithmic potential, $V(s) = c_1 \ln r + c_2$. When the two constants are determined to yield $V(a) = 0$ and $V(b) = V$ we have

$$V(r) = V \frac{\ln(r/a)}{\ln(b/a)}.$$

The gain in potential energy, $sV(r_m)$, is converted into kinetic energy:

$$\frac{1}{2} mv^2 = eV \frac{\ln(r_m/a)}{\ln(b/a)}.$$

Thus

$$v = \sqrt{\frac{2eV}{m} \frac{\ln(r_m/a)}{\ln(b/a)}}. \tag{5}$$

(4) and (5) seem to be different answers. This is only apparent since r_m is not an independent parameter, but determined by B and V so that the two answers are identical.

e) For the critical magnetic field the maximal distance r_m equals b, the radius of the outer cylinder, and the speed at the turning point is then

$$v = \frac{eB(b^2 - a^2)}{2mb}.$$

Since the Lorentz force does no work, the corresponding kinetic energy $\frac{1}{2}mv^2$ equals eV (question a):

$$v = \sqrt{2eV/m}$$

The last two equations are consistent when

$$\frac{eB(b^2 - a^2)}{2mb} = \sqrt{2eV/m}.$$

The critical magnetic field for current cut-off is therefore

$$\underline{\underline{B_c = \frac{2b}{b^2 - a^2}\sqrt{\frac{2mV}{e}}}}.$$

f) The Lorentz force has no component parallel to the magnetic field, and consequently the velocity component v_B is constant under the motion. The corresponding displacement parallel to the cylinder axis has no relevance for the question of reaching the anode.

Let v denote the final azimuthal speed of an electron that barely reaches the anode. Conservation of energy implies that

$$\tfrac{1}{2}m(v_B^2 + v_\phi^2 + v_r^2) + eV = \tfrac{1}{2}m(v_B^2 + v^2),$$

giving

$$v = \sqrt{v_r^2 + v_\phi^2 + 2eV/m}. \qquad (6)$$

Evaluating the constant C in (3) at both cylinder surfaces for the critical situation we have

$$mv_\phi a - \tfrac{1}{2}eB_c a^2 = mvb - \tfrac{1}{2}eB_c b^2.$$

Insertion of the value (6) for the velocity v yields the critical field

$$B_c = \frac{2m(vb - v_\phi a)}{e(b^2 - a^2)} = \frac{2mb}{e(b^2 - a^2)}\left[\sqrt{v_r^2 + v_\phi^2 + 2eV/m} - v_\phi a/b\right].$$

Solution Problem 3

a) With the centre of the earth as origin, let the centre of mass C be located at $\vec{l}$. The distance l is determined by

$$M\,l = M_m\,(L - l),$$

which gives

$$l = \frac{M_m}{M + M_m} L = \underline{4.63 \cdot 10^6\ \text{m},} \qquad (1)$$

less than R, and thus inside the earth.

The centrifugal force must balance the gravitational attraction between the moon and the earth: 113

$$M\omega^2 l = G\frac{MM_m}{L^2},$$

which gives

$$\omega = \sqrt{\frac{GM_m}{L^2 l}} = \underline{\underline{\sqrt{\frac{G(M + M_m)}{L^3}}}} = \underline{\underline{2.67 \cdot 10^{-6}\,\text{s}^{-1}}}. \qquad (2)$$

(This corresponds to a period $2\pi/\omega = 27.2$ days.) We have used (1) to eliminate l.

b) The potential energy of the mass point m consists of three contributions:

(1) Potential energy because of rotation (in the rotating frame of reference, see the problem text),

$$-\frac{1}{2}m\omega^2 r_1^2,$$

where $\vec{r}_1$ is the distance from C. This corresponds to the centrifugal force $m\omega^2 r_1$, directed outwards from C.

(2) Gravitational attraction to the earth,

$$-G\frac{mM}{r}.$$

(3) Gravitational attraction to the moon,