Theory Question 3: Solution

Solar-Powered Aircraft

(a) The force $\vec{F}$ required to change the velocity $\Delta\vec{v}$ of a fluid whose flow rate is $\frac{dm}{dt}$ is given by:

$$\vec{F} = \Delta\vec{v}\ \frac{dm}{dt}$$

In this case the flow rate is:

$$\frac{dm}{dt} = x\, l \rho\, v = \frac{\pi}{4} l^2\ \rho\, v$$

The vertical component of $\Delta\vec{v}$ is:

$$\Delta v_V = v \sin\varepsilon$$

The horizontal component of $\Delta\vec{v}$ is:

$$\Delta\, v_H = v\,(1 - \cos\varepsilon)$$

We can now write expressions for the lift L and drag D_1 (called form drag).

$$\underline{L = \frac{\pi}{4} \rho\, v^2 l^2 \sin\varepsilon}$$

$$\underline{D_1 = \frac{\pi}{4} \rho\, v^2 l^2 (1 - \cos\varepsilon\,)}$$

Approximations using ε instead of sin ε etc. are allowed.

(b) The power required to keep the aircraft flying straight and level is given by:

$$P = Dv = (D_1 + D_2)v$$

The horizontal drag force D_2 (frictional drag) is given by the rate of change of momentum of the air flowing past the wing due to friction:

$$D_2 = v_1 \frac{dm_1}{dt} - v_2 \frac{dm_2}{dt}$$

Since the wing is neither a source nor a sink, the mass flow of air into the wing $\left(\frac{dm_1}{dt}\right)$ must be the same as the mass flow from the wing $\left(\frac{dm_2}{dt}\right)$ therefore:

$$\frac{dm_1}{dt} = \frac{dm_2}{dt} = \frac{dm}{dt} = xl\rho v$$

Substituting $v_1=v$ and $v_2=v-\Delta v$:

$$D_2 = vxl\rho v - (v-\Delta v)xl\rho\, v = xl\rho v^2 - xl\rho v^2 + v\Delta vxl\rho = xl\rho\, v\Delta v = \frac{\pi l}{4} l\rho\, v\Delta v = \frac{\pi f}{4A}\rho\, v^2 l^2$$

(This drag is necessarily along the wing surface; when the wing is at an angle ε, the horizontal component is this value multiplied by $\cos\varepsilon$.:

$$\frac{\pi f}{4A}\rho\, v^2 l^2 \cos\varepsilon \approx \frac{\pi f}{4A}\rho\, v^2 l^2\left(1-\frac{\varepsilon^2}{2}\right) \approx \frac{\pi f}{4A}\rho\, v^2 l^2 \quad + O\left(\varepsilon^2 f\right)$$

so to the order given, our simple answer is correct.)

The total drag force $D=D_1+D_2$ is dependent on the deflection angle ε and the drag coefficient f:

$$D = \frac{\pi}{4}\rho\, v^2 l^2\left((1-\cos\varepsilon) + \frac{f}{A}\right) \approx \frac{\pi}{4}\rho\, v^2 l^2\left(\frac{1}{2}\sin^2\varepsilon + \frac{f}{A}\right)$$

In making this approximation D can be expressed in terms of the mass, speed and wing dimensions of the aircraft. Note that for level flight the lift has to be equal to the weight of the craft.

$$L = Mg = \frac{\pi}{4}\rho v^2 l^2 \sin\varepsilon \;; \quad \sin\varepsilon = \frac{4Mg}{\pi\rho v^2 l^2}$$

We can now minimize power with respect to either v or ε; here we choose v.

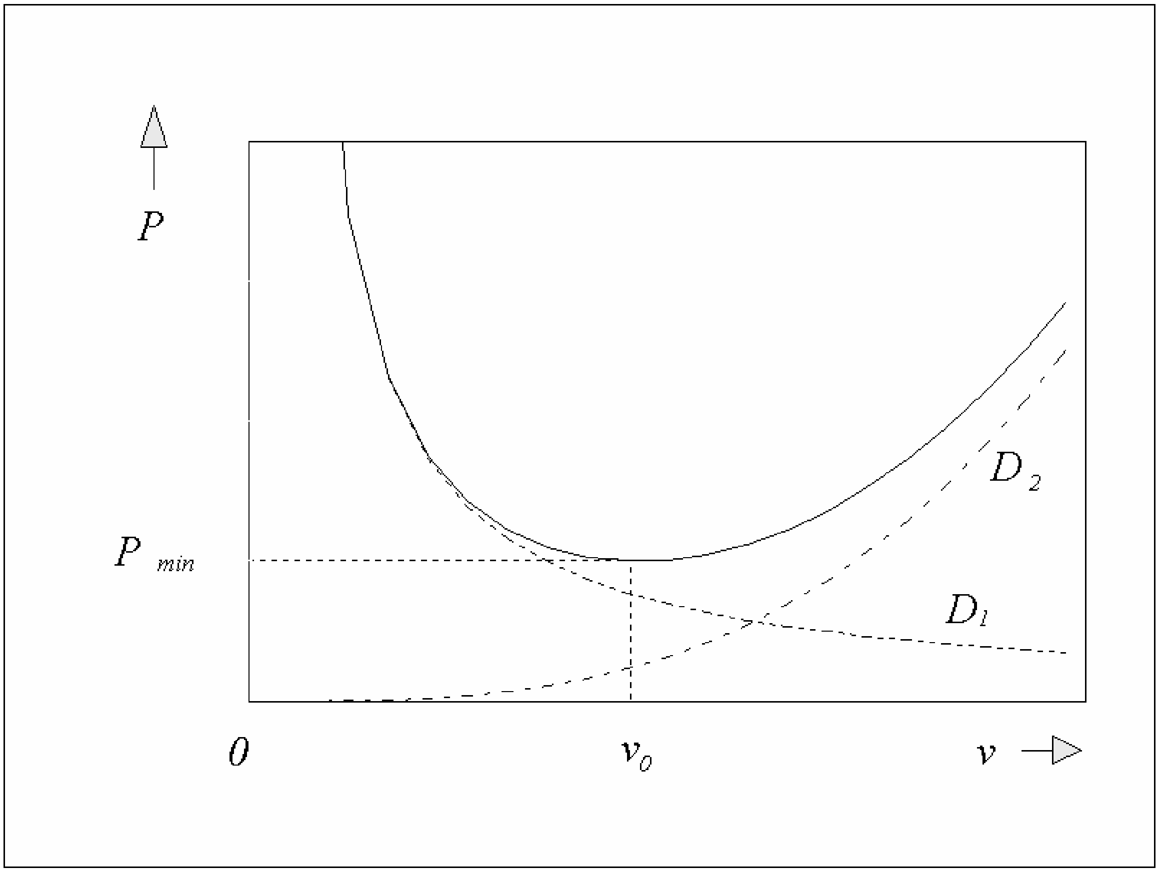
$$P = Dv = \frac{\pi}{4}\rho v^3 l^2 \left(\frac{f}{A} + \frac{1}{2}\frac{\left(4Mg\right)^2}{\left(\pi\rho v^2 l^2\right)^2} \right) = \frac{\pi}{4}\rho v^3 l^2 \ \frac{f}{A} + \frac{2\left(Mg\right)^2}{\pi\rho v l^2}$$

$$\frac{dP}{dv} = \frac{3\pi}{4}\rho v^2 l^2 \ \frac{f}{A} - \frac{2\left(Mg\right)^2}{\pi\rho v^2 l^2} = 0 \text{, when } v = v_0$$

Flight velocity for minimum power:

$$v_0^{\ 4} = \frac{8\left(Mg\right)^2 A}{3\pi^2 \rho^2 l^4 f} = \frac{8}{3Af}\left(\frac{Mg}{\pi\,\rho\, S} \right)^2$$

(c) The graph of power vs. velocity is as follows:



$$P_{\min} = \frac{\pi}{4}\rho\, {v_0}^3 l^2 \left(\frac{f}{A} + \frac{1}{2}\frac{(4Mg)^2}{\left(\pi\rho v_0^2\, l^2\right)^2} \right) = \frac{\pi}{4}\rho\, {v_0}^3 l^2 \left(\frac{f}{A} + \frac{(4Mg)^2}{2\left(\pi\rho l^2\right)^2}\frac{3\pi^2\rho^2 l^4 f}{8(Mg)^2 A} \right)$$

$$= \pi\rho {v_0}^3 l^2 \frac{f}{A} = \pi\rho {v_0}^3 Sf$$

Substitute for v_0 :

$$P_{\min} = \pi\rho S f \frac{8^{\frac{3}{4}}(Mg)^{\frac{3}{2}}}{(3Af)^{\frac{3}{4}}(\pi\rho S)^{\frac{3}{2}}} = \left(\frac{8}{3A}\right)^{\frac{3}{4}} f^{\frac{1}{4}} \frac{(Mg)^{\frac{3}{2}}}{(\pi\rho S)^{\frac{1}{2}}}$$

(d) Equate this to available power, $P_{avail} = IS = P_{\min}$:

$$\left(\frac{Mg}{S}\right)^{\frac{3}{2}} = I\left(\frac{3A}{8}\right)^{\frac{3}{4}} \frac{\left(\pi\rho\right)^{\frac{1}{2}}}{f^{\frac{1}{4}}}$$

$$\frac{Mg}{S} = I^{\frac{2}{3}}\left(\frac{3A}{8}\right)^{\frac{1}{2}} \frac{\left(\pi\rho\right)^{\frac{1}{3}}}{f^{\frac{1}{6}}}$$

The numerical answers are:

$$\frac{Mg}{S} = 35.6 \text{ N/m}^2 , \quad v_0 = 8.60 \text{ m/s}$$

Theory Question No.3: Mark Distribution

Smallest fractional mark allowed: 0.25

Marks allowed for errors consistently propagated only if physically reasonable.

	MAXIMUM	SCORE	SUBTOTAL
(a) Formulation of Newton II	1		
Approach to drag	.5		
D_1 formula:	.5		
Approach to lift	.5		
L formula:	.5		
			(a)
(b) Correct approach to D_2	.5		
Correct expression for D_2	.5		
Correct minimization approach	1		
Expression for v_0	1		
			(b)
(c) Correct expression for power	1		
Graph ($1/v$, v^3 forms)	1		
			(c)
(d) Correct wing loading	1		
Correct speed	1		
			(d)
TOTAL	10		Total

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