2 Water under an ice cap[6]

2.1 Problem text

An ice cap is a thick sheet of ice (up to a few km in thickness) resting on the ground below and extending horizontally over tens or hundreds of km. In this problem we consider the melting of ice and the behavior of water under a temperate ice cap, i.e. an ice cap at the melting point. We may assume that under such conditions the ice causes pressure variations as a viscous fluid, but deforms in a brittle fashion, principally by vertical movement. For the purposes of this problem the following information is given.

Density of water:	$\rho_w = 1.000 \cdot 10^3$ kg/m^3
Density of ice:	$\rho_i = 0.917 \cdot 10^3$ kg/m^3
Specific heat of ice:	$c_i = 2.1 \cdot 10^3$ J/(kg °C)
Specific latent heat of ice:	$L_i = 3.4 \cdot 10^5$ J/kg
Density of rock and magma:	$\rho_r = 2.9 \cdot 10^3$ kg/m^3
Specific heat of rock and magma:	$c_r = 700$ J/(kg °C)
Specific latent heat of rock and magma:	$L_r = 4.2 \cdot 10^5$ J/kg
Average outward heat flow through the surface of the earth:	$J_Q = 0.06$ W/m^2
Melting point of ice:	$T_0 = 0$°C, constant

a) (0.5 points) Consider a thick ice cap at a location of average heat flow from the interior of the earth. Using the data from the table, calculate the thickness d of the ice layer melted every year and write your answer in the designated box on the answer sheet.

b) (3.5 points) Consider now the upper surface of an ice cap. The ground below the ice cap has a slope angle α. The upper surface of the cap slopes by an angle β as shown in Figure 2.1. The vertical thickness of the ice at $x = 0$ is h_0. Hence the lower and upper surfaces of the ice cap can be described by the equations

$$y_1 = x \tan\alpha, \quad y_2 = h_0 + x \tan\beta \tag{2.1}$$

Derive an expression for the pressure p at the bottom of the ice cap as a function of the horizontal coordinate x and write it on the answer sheet.

Formulate mathematically a condition between β and α, so that water in a layer between the ice cap and the ground will flow in neither direction. Show that the condition is of the form $\tan\beta = s\tan\alpha$. Find the coefficient s and write the result in a symbolic form on the answer sheet.

The line $y_1 = 0.8\,x$ in Figure 2.2 shows the surface of the earth below an ice cap. The vertical thickness h_0 at $x = 0$ is 2 km. Assume that water at the bottom is in equilibrium.

On a graph answer sheet draw the line y_1 and add a line y_2 showing the upper surface of the ice. Indicate on the figure which line is which.

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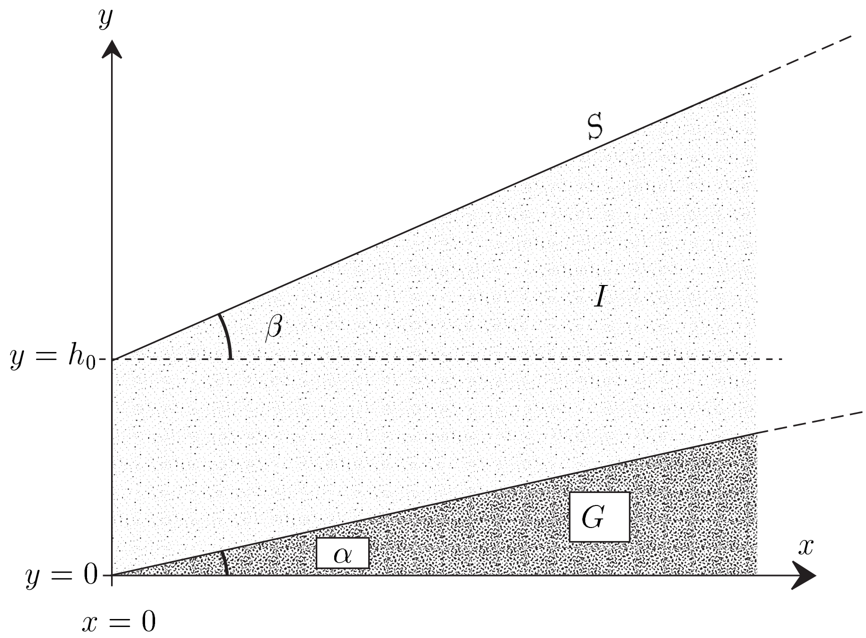

Figure 2.1: *Cross section of an ice cap with a plane surface resting on an inclined plane ground. S: surface, G: ground, I: ice cap.*

c) (1 point) Within a large ice sheet on horizontal ground and originally of constant thickness $D = 2.0$ km, a conical body of water of height $H = 1.0$ km and radius $r = 1.0$ km is formed rather suddenly by melting of the ice (Figure 2.3). We assume that the remaining ice adapts to this by vertical motion only.

Show analytically on a blank answer sheet and pictorially on a graph answer sheet, the shape of the surface of the ice cap after the water cone has formed and hydrostatic equilibrium has been reached.

d) (5 points) In its annual expedition an international group of scientists explores a temperate ice cap in Antarctica. The area is normally a wide plateau but this time they find a deep crater-like depression, formed like a top-down cone with a depth h of 100 m and a radius r of 500 m (Figure 2.4). The thickness of the ice in the area is 2000 m.

After a discussion the scientists conclude that most probably there was a minor volcanic eruption below the ice cap. A small amount of magma (molten rock) intruded at the bottom of the ice cap, solidified and cooled, melting a certain volume of ice. The scientists try as follows to estimate the volume of the intrusion and get an idea of what became of the melt water.

Assume that the ice only moved vertically. Also assume that the magma was completely molten and at 1200°C at the start. For simplicity, assume further that the intrusion had the form of a cone with a circular base vertically below the conical depression in the surface. The time for the rising of the magma was short relative to the time for the exchange of heat in the process. The heat flow is assumed to have been primarily vertical such that the volume melted from the ice at any time is bounded by a conical surface centered above the center of the magma intrusion.

Given these assumptions the melting of the ice takes place in two steps. At first the water is not in pressure equilibrium at the surface of the magma and hence flows away. The water flowing away can be assumed to have a temperature of 0°C. Subsequently,

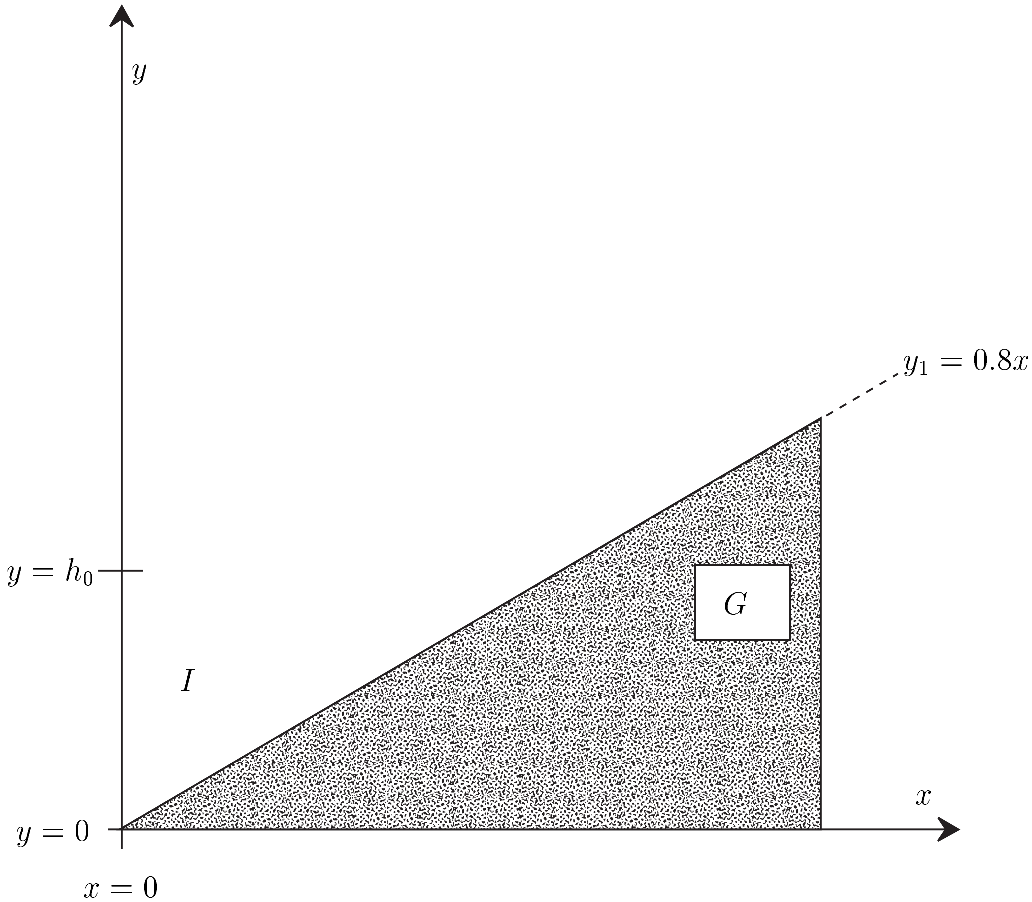

Figure 2.2: *Cross section of a temperate ice cap resting on an inclined ground with water at the bottom in equilibrium. G: ground, I: ice cap.*

hydrostatic equilibrium is reached and the water accumulates above the intrusion instead of flowing away.

When thermal equilibrium has been reached, you are asked to determine the following quantities. Write the answers on the answer sheet.

1. The height H of the top of the water cone formed under the ice cap, relative to the original bottom of the ice cap.
2. The height h_1 of the intrusion.
3. The total mass m_{tot} of the water produced and the mass m' of water that flows away.

Plot on a graph answer sheet, to scale, the shapes of the rock intrusion and of the body of water remaining. Use the coordinate system suggested in Figure 2.4.

2.2 Solution

a)

Based on the conservation of energy we have

$$J_Q \cdot 1 \text{ year} = L_i \rho_i d \tag{2.2}$$

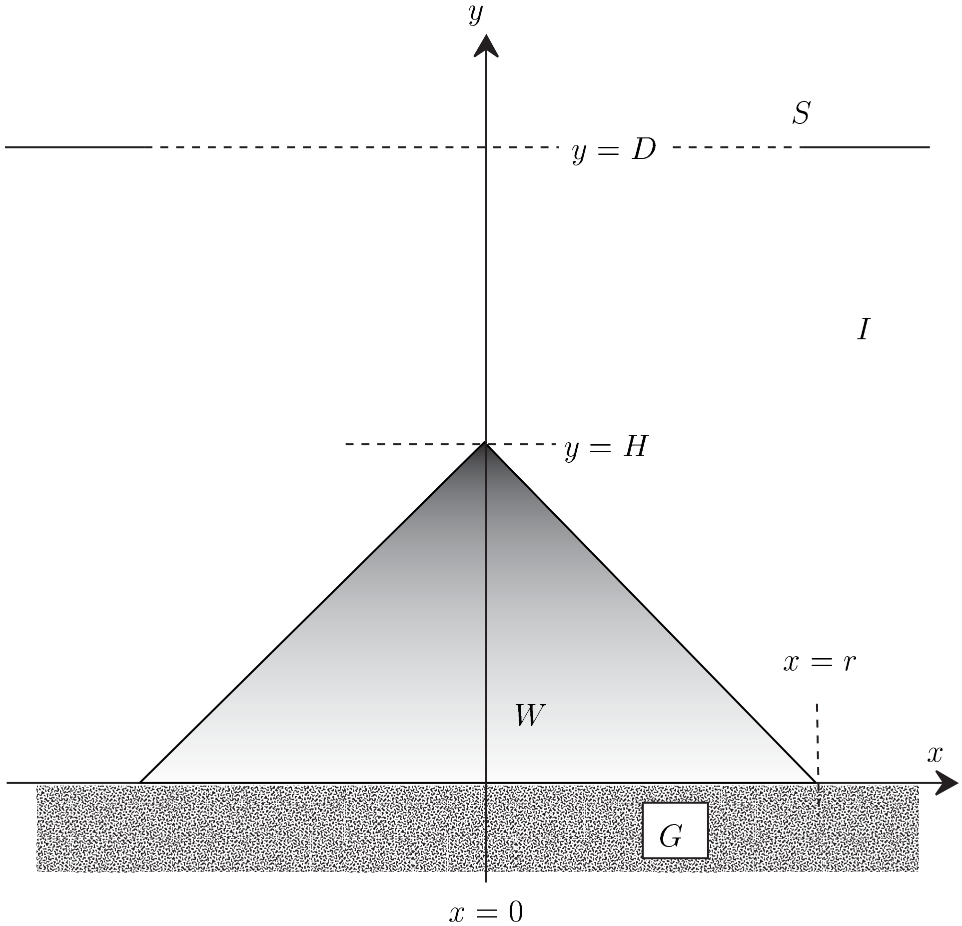

Figure 2.3: *A vertical section through the mid-plane of a water cone inside an ice cap. S: surface, W: water, G: ground, I: ice cap.*

$$\boldsymbol{d} = \frac{J_Q \cdot 1 \text{ year}}{L_i \ \rho_i} = \frac{0.06 \text{ J s}^{-1} \text{ m}^{-2} \ 365.25 \ \cdot 24 \ \cdot 60 \ \cdot 60 \text{ s}}{3.4 \cdot 10^5 \text{ J/kg } 917 \text{ kg/m}^3} = \mathbf{6.1 \cdot 10^{-3} m} \tag{2.3}$$

b)

Let p_a be the atmospheric pressure, taken to be constant. At a depth z inside the ice cap the pressure is given by:

$$p = \rho_i g z + p_a \tag{2.4}$$

Therefore, at the bottom of the ice cap, where $z = y_2 - y_1$:

$$\boldsymbol{p} = \rho_i g (y_2 - y_1) + p_a \tag{2.5}$$

$$= \boldsymbol{\rho_i g x (\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a} \tag{2.6}$$

For water not to move at the base of the ice cap the pressure must be hydrostatic (trivial, but can be seen from Bernoulli's equation), i.e.

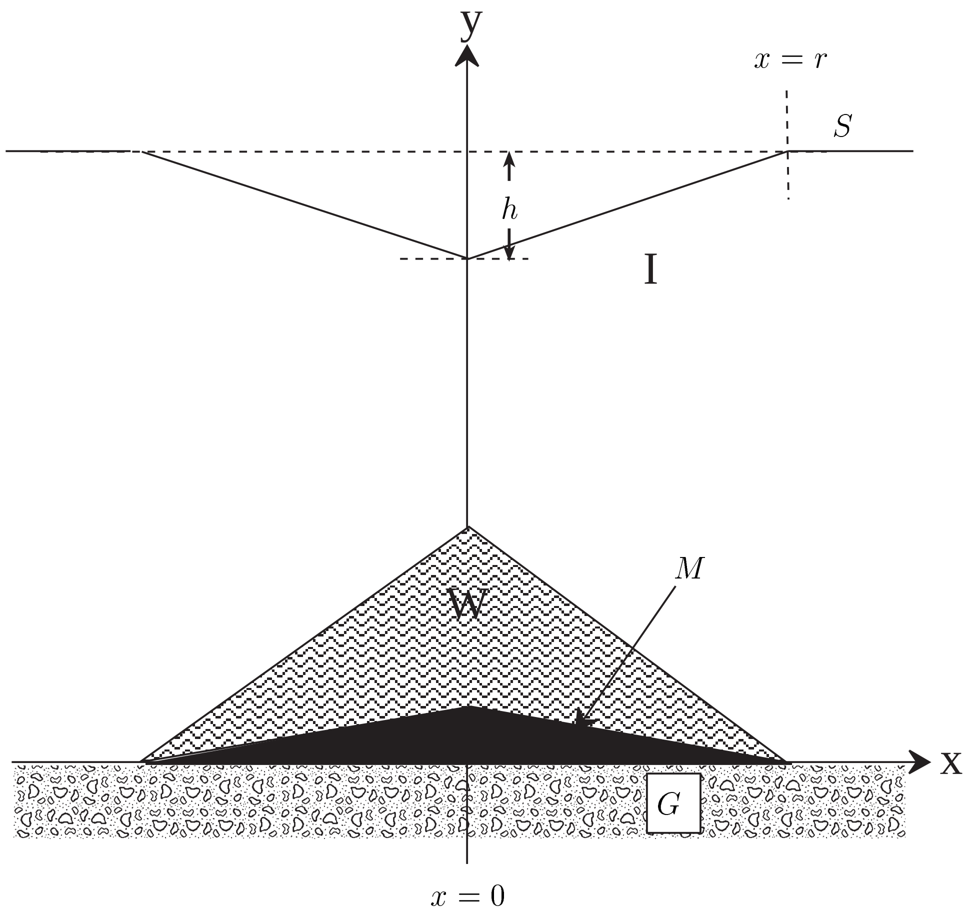

Figure 2.4: *A vertical and central cross section of a conical depression in a temperate ice cap. S: surface, G: ground, I: ice cap, M: rock/magma intrusion, W: water. Note that the figure is NOT drawn to scale.*

$$p = constant - \rho_w g y_1 \quad (2.7)$$
$$= constant - \rho_w g x \tan\alpha \quad (2.8)$$

Therefore

$$\rho_i g x(\tan\beta - \tan\alpha) = -\rho_w g x \tan\alpha \quad (2.9)$$

leading to

$$\mathbf{tan}\,\boldsymbol{\beta} = -\frac{\rho_w - \rho_i}{\rho_i}\tan\alpha = \boldsymbol{-\frac{\Delta\rho}{\rho_i}}\ \mathbf{tan}\,\boldsymbol{\alpha} \approx -0.091\ \tan\alpha \quad (2.10)$$
$$\boldsymbol{s = -\Delta\rho/\rho_i = -0.091} \quad (2.11)$$
$$(2.12)$$

where the minus-sign is significant.

This can also be seen in various ways by looking at a mass element of water at the bottom of the ice and demanding equilibrium. – We now proceed with the solution.

With $\tan\alpha = 0.8$, we get $\tan\beta = -0.073$ and

$$\boldsymbol{y_2 = 2 \text{ km} - 0.073\, x} \tag{2.13}$$

The students are supposed to draw this line on a graph.

c)

Since the ice adapts by vertical motion only we see that the conical depression at the surface will have the same radius of 1.0 km as the intrusion. According to (b) it will have a depth of

$$h = |r\tan\beta| = \frac{\Delta\rho}{\rho_i} r\tan\alpha \tag{2.14}$$

$$= \frac{\Delta\rho}{\rho_i} H \tag{2.15}$$

$$= 0.091 \cdot 1 \text{ km} = 91 \text{ m}. \tag{2.16}$$

The students are supposed to show this result as a graph.

d)

The volume of a circular cone is $V = \frac{1}{3}\pi r^2 h$. We assume that the height of the intrusion is h_1. We may say that it firstly melts an ice cone of its own volume $V_1 = \frac{1}{3}\pi r^2 h_1$. Pressure equilibrium has not yet been reached. Hence the water will flow away and the ice will keep contact with the face of the intrusion making the upper surface of the ice horizontal again. The intrusion then melts a volume equivalent to a cone of height $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ whereupon pressure equilibrium has been reached (following part (c)). During this second phase the melted water will also flow away. Assuming that the intrusion still has not cooled down to 0°C the intrusion will further melt a volume equivalent to a cone of height h_3, its water accumulating in place, forming a cone of height $h_3' = \frac{\rho_i}{\rho_w} h_3$ relative to the top of the intrusion. The total height of the ice cone melted is

$$h_{tot} = h_1 + h_2 + h_3 \tag{2.17}$$

The depth of the depression at the surface will be given by

$$h = \frac{\Delta\rho}{\rho_i}\,(h_1 + h_3') \tag{2.18}$$

which is most easily seen by considering pressure equilibrium in the final situation (again following part (c)). Thus, the requested height of the top of the water cone is

$$\boldsymbol{H} = h_1 + h_3' = \frac{\rho_i}{\Delta\rho}\, h = \mathbf{1.1 \times 10^3 m} \tag{2.19}$$

The heat balance gives

$$\frac{1}{3}\pi\, r^2\, \{\rho_r h_1 (L_r + c_r\, \Delta T) - \rho_i\, L_i\, h_{tot}\} = 0 \tag{2.20}$$

where $\Delta T = 1200°C$ is the change in temperature of the rock intrusion. Following equation (2.17) and using the facts that $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ and $h_3 = \frac{\rho_w}{\rho_i} h'_3$ we obtain

$$h_{tot} = h_1 + \frac{\Delta\rho}{\rho_i} h_1 + \frac{\rho_w}{\rho_i} h'_3 = \frac{\rho_w}{\rho_i} (h_1 + h'_3) \tag{2.21}$$

Therefore (using equation (2.19))

$$h_{tot} = \frac{\rho_w}{\rho_i} (h_1 + h'_3) = \frac{\rho_w}{\rho_i} \ H = \frac{\rho_w}{\Delta\rho} \ h = 1.20 \cdot 10^3 \text{m} \tag{2.22}$$

This implies that the cone does not reach the surface of the ice cap. Inserting the result into the equation (2.20) we can solve for h_1:

$$\rho_r \ h_1 \ (L_r + \ c_r \ \Delta T) = \frac{\rho_i \ \rho_w \ L_i \ h}{\Delta\rho} \tag{2.23}$$

$$\boldsymbol{h_1} = \frac{\rho_i \ \rho_w \ L_i \ h}{\Delta\rho \ \rho_r \ (L_r + \ c_r \ \Delta T)} \tag{2.24}$$

$$= \mathbf{103 \ m} \tag{2.25}$$

The total mass of water formed is of course equal to the mass of the ice melted and is

$$\boldsymbol{m_{tot}} = \rho_i \ (1/3) \ \pi \ r^2 \ h_{tot} = \mathbf{2.9 \cdot 10^{11} \ kg} \tag{2.26}$$

The mass of the water which flows away is

$$\boldsymbol{m'} = \frac{h_1 + h_2}{h_{tot}} \ m_{tot} = \frac{\rho_w \ h_1}{\rho_i \ h_{tot}} \ m_{tot} = \mathbf{2.7 \cdot 10^{10} \ kg} \tag{2.27}$$

The students are finally expected to plot the shapes of the rock intrusion and the water body.

2.3 Grading scheme

2(a)	
Answer: equation (2.3), $d = 6.1 \cdot 10^{-3}$ m	**0.5**

2(b)	
Answer i): equation (2.6): $p = \rho_i g x(\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a$	**1.0**
Answer ii): equation (2.10): $s = -\frac{\rho_w - \rho_i}{\rho_i} = -\frac{\Delta\rho}{\rho_i}$	**2.0**
Answer iii): Graph based on equation (2.13)	**0.5**

2(c)	
Answer: Depth, radius and graph, $r = 1000$ m, $h = 91$ m	**1.0**

2(d)	
Answer i): Height of water cone as in (2.19): $H = 1.1 \cdot 10^3$ m	**2.0**
Answer ii): Height of intrusion as in (2.25): $h_1 = 103$ m	**1.0**
Answer iii): Total mass of melt water as in (2.26): $m_{tot} = 2.9 \cdot 10^{11}$ kg	**0.5**
Answer iv): Mass of water flowing away as in (2.27): $m' = 2.7 \cdot 10^{10}$ kg	**1.0**
Answer v): Graph	**0.5**