3 Faster than light?[7]

3.1 Problem text

In this problem we analyze and interpret measurements made in 1994 on radio wave emission from a compound source within our galaxy.

The receiver was tuned to a broad band of radio waves of wavelengths of several centimeters. Figure 3.1 shows a series of images recorded at different times. The contours indicate constant radiation strength in much the same way as altitude contours on a geographical map. In the figure the two maxima are interpreted as showing two objects moving away from a common center shown by crosses in the images. (The center, which is assumed to be fixed in space, is also a strong radiation emitter but mainly at other wavelengths). The measurements conducted on the various dates were made at the same time of day.

The scale of the figure is given by a line segment showing one arc second (as). (1 as = 1/3600 of a degree). The distance to the celestial body at the center of the figure, indicated by crosses, is estimated to be $R = 12.5$ kpc. A kiloparsec (kpc) equals $3.09 \cdot 10^{19}$ m. The speed of light is $c = 3.00 \cdot 10^8$ m/s. Error calculations are not required in the solution.

a) (2 points) We denote the angular positions of the two ejected radio emitters, relative to the common center, by $\theta_1(t)$ and $\theta_2(t)$, where the subscripts 1 and 2 refer to the left and right hand ones, respectively, and t is the time of observation. The angular speeds, as seen from the Earth, are ω_1 and ω_2. The corresponding apparent transverse linear speeds of the two sources are denoted by $v'_{1,\perp}$ and $v'_{2,\perp}$.

Using Figure 3.1, make a graph to find the numerical values of ω_1 and ω_2 in milli-arc-seconds per day (mas/d). Also determine the numerical values of $v'_{1,\perp}$ and $v'_{2,\perp}$, and write all answers on the answer sheet. (You may be puzzled by some of the results).

b) (3 points) In order to resolve the puzzle arising in part (a), consider a light-source moving with velocity $\vec{v}$ at an angle ϕ $(0 \leq \phi \leq \pi)$ to the direction towards a distant observer O (Figure 3.2). The speed may be written as $v = \beta c$, where c is the speed of light. The distance to the source, as measured by the observer, is R. The angular speed of the source, as seen from the observer, is ω, and the apparent linear speed perpendicular to the line of sight is $v'_\perp$.

Find ω and $v'_\perp$ in terms of β, R and ϕ and write your answer on the answer sheet.

c) (1 point) We assume that the two ejected objects, described in the introduction and in part (a), are moving in opposite directions with equal speeds $v = \beta c$. Then the results of part (b) make it possible to calculate β and ϕ from the angular speeds ω_1 and ω_2 and the distance R. Here ϕ is the angle defined in part (b), for the left hand object, corresponding to subscript 1 in part (a).

Derive formulas for β and ϕ in terms of known quantities and determine their numerical values from the data in part (a). Write your answers in the designated fields on the answer sheet.

d) (2 points) In the one-body situation of part (b), find the condition for the apparent perpendicular speed $v'_\perp$ to be larger than the speed of light c.

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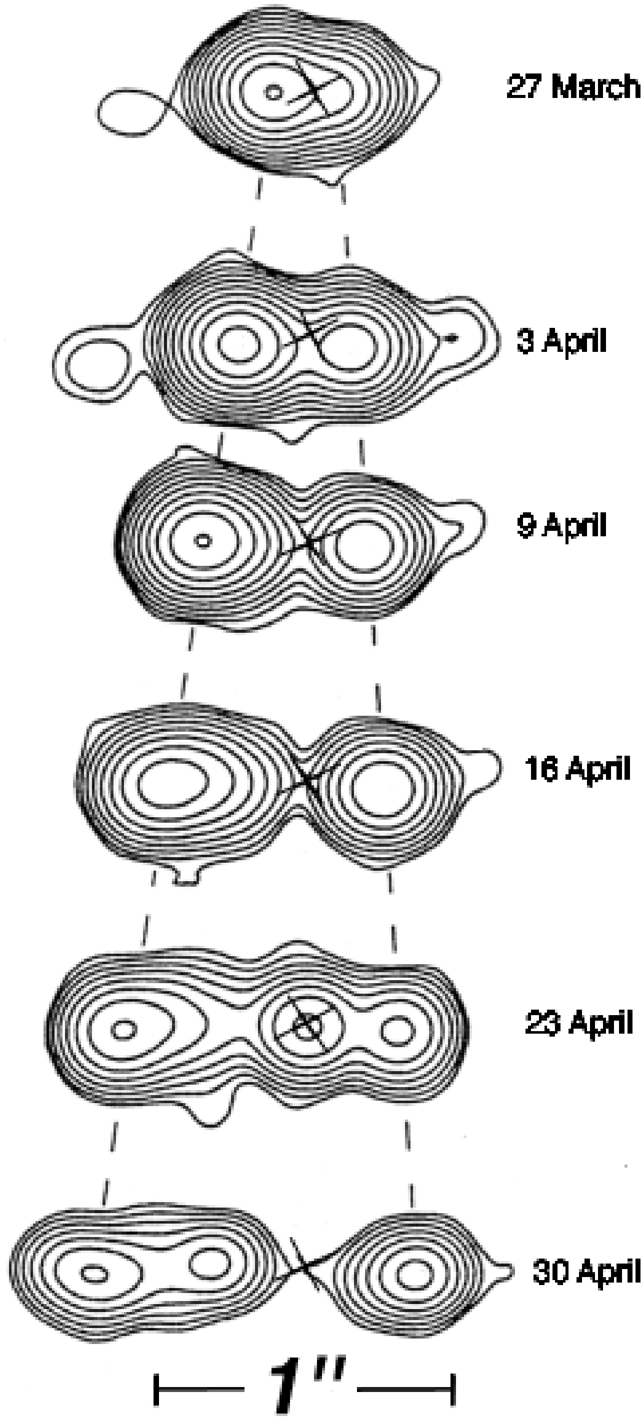


Figure 3.1: *Radio emission from a source in our galaxy.*

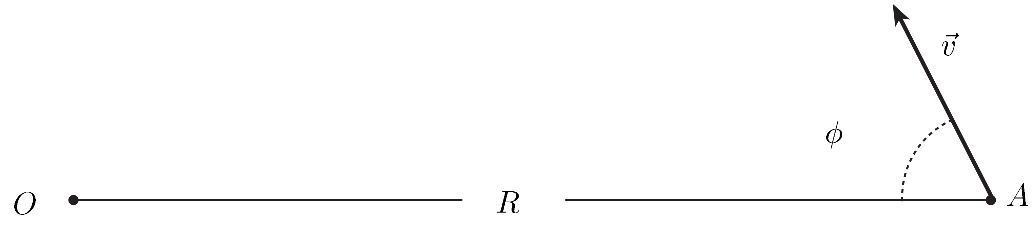


Figure 3.2: *The observer is at O and the original position of the light source is at A. The velocity vector is $\vec{v}$.*

Write the condition in the form $\beta > f(\phi)$ and provide an analytic expression for the function f on the answer sheet.

Draw on the graph answer sheet the physically relevant region of the (β, ϕ)-plane. Show by shading in which part of this region the condition $v'_\perp > c$ holds.

e) (1 point) Still in the one-body situation of part (b), find an expression for the maximum value $(v'_\perp)_{max}$ of the apparent perpendicular speed $v'_\perp$ for a given β and write it in the designated field on the answer sheet. Note that this speed increases without limit when $\beta \to 1$.

f) (1 point) The estimate for R given in the introduction is not very reliable. Scientists have therefore started speculating on a better and more direct method for determining R. One idea for this goes as follows. Assume that we can identify and measure the Doppler shifted wavelengths λ_1 and λ_2 of radiation from the two ejected objects, corresponding to the same known original wavelength λ_0 in the rest frames of the objects.

Starting from the equations for the relativistic Doppler shift, $\lambda = \lambda_0(1 - \beta \cos\phi)(1 - \beta^2)^{-1/2}$, and assuming, as before, that both objects have the same speed, v, show that the unknown $\beta = v/c$ can be expressed in terms of λ_0, λ_1, and λ_2 as

$$\beta = \sqrt{1 - \frac{\alpha\,\lambda_0^2}{(\lambda_1 + \lambda_2)^2}}\,. \tag{3.1}$$

Write the numerical value of the coefficient α in the designated field on the answer sheet.

You may note that this means that the suggested wavelength measurements will in practice provide a new estimate of the distance.

3.2 Solution

a) On Figure 3.1 we mark the centers of the sources as neatly as we can. Let $\theta_1(t)$ be the angular distance of the left center from the cross as a function of time and $\theta_2(t)$ the angular distance of the right center. We measure these quantities on the figure at the given times by a ruler and convert to arcseconds according to the given scale. This results in the following numerical data:

time [days]	θ_1 [as]	θ_2 [as]
0	0.139	0.076
7	0.253	0.139
13	0.354	0.190
20	0.468	0.253
27	0.601	0.316
34	0.709	0.367

The uncertainty in the readings by the ruler is estimated to be ± 0.5 mm, resulting in the uncertainty of $\pm$ 0.013 as in the θ values. We plot the data in Figure 3.3.

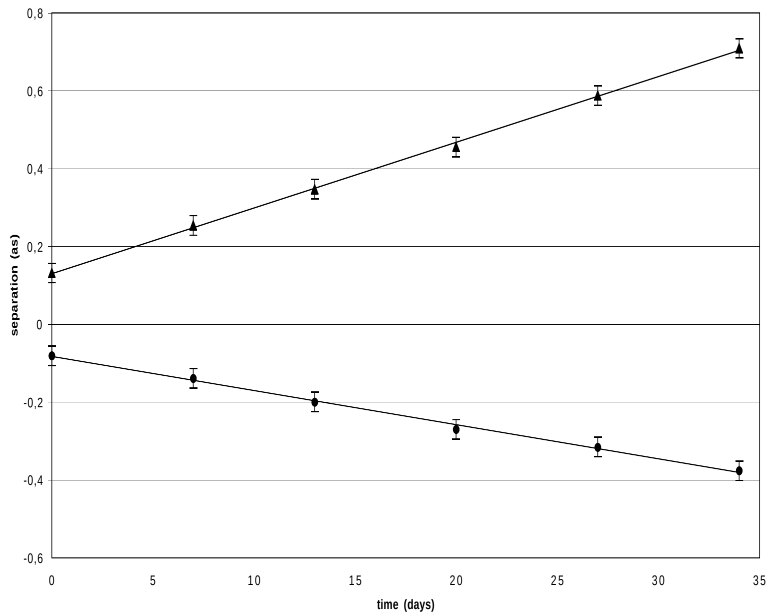


Figure 3.3: *The angular distances θ_1 and θ_2 (in as) as functions of the time in days.*

Fitting straight lines through the data results in:

$$\boldsymbol{\omega_1} = \mathrm{d}\theta_1/\mathrm{dt} = \mathbf{(17.0 \pm 1.0)\ mas/day} = 9.54 \cdot 10^{-13}\mathrm{rad/s} \quad (3.2)$$

$$\boldsymbol{\omega_2} = \mathrm{d}\theta_2/\mathrm{dt} = \mathbf{(8.7 \pm 1.0)\ mas/day} = 4.88 \cdot 10^{-13}\mathrm{rad/s} \quad (3.3)$$

$$\boldsymbol{v'_{1,\perp}} = \omega_1\ R = 9.54 \cdot 10^{-13} \cdot 12.5 \cdot 3.09 \cdot 10^{19} \quad (3.4)$$

$$= \mathbf{3.68 \cdot 10^8 m/s \approx (1.23 \pm 0.07)}\ \boldsymbol{c} \quad (3.5)$$

$$\boldsymbol{v'_{2,\perp}} = \mathbf{1.89 \cdot 10^8 m/s \approx (0.63 \pm 0.07)}\ \boldsymbol{c} \quad (3.6)$$

b) We consider the motion of the source during the time interval Δt from the point A to the point A', see Figure 3.4.

We then have

$$\vec{r}_{AA'} = \vec{r}_{A'} - \vec{r}_A = \vec{v} \cdot \Delta t\ . \quad (3.7)$$

Now let $\Delta t'$ denote the difference in arrival times at O of the signals from A and A'. Due to the different distances to A and A' and the finite speed of light, c, we have

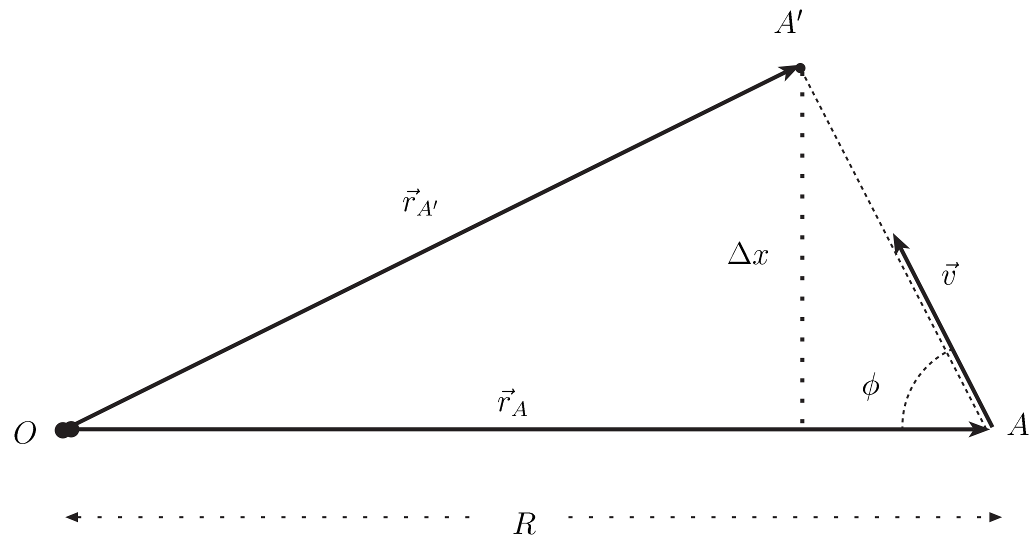


Figure 3.4: *The observer is at O and the original position of the source is at A. The velocity vector is $\vec{v}$.*

$$\Delta t' = \Delta t + (r_{A'} - r_A)/c \; . \tag{3.8}$$

For small Δt, such that $v\ \Delta t << r_A = R$, we have

$$r_{A'} - r_A \approx -v\ \Delta t \cos\phi \tag{3.9}$$

and hence

$$\Delta t' \approx \Delta t\ (1 - \beta\cos\phi)\ ;\ \ \beta = v/c \; . \tag{3.10}$$

This implies that an observer at O will find the apparent transverse speed of the source to be

$$\boldsymbol{v'_\perp} = \frac{\Delta x}{\Delta t'} = \frac{\Delta x}{\Delta t\ (1 - \beta\cos\phi)} = \frac{\boldsymbol{c\beta\sin\phi}}{\boldsymbol{1 - \beta\cos\phi}} \tag{3.11}$$

where we have used that the real transverse speed in the reference frame of the observer is $v_\perp = \Delta x/\Delta t = c\beta\sin\phi$.

The angular speed observed at O is

$$\boldsymbol{\omega} = \frac{v'_\perp}{R} = \frac{\boldsymbol{c\beta\sin\phi}}{\boldsymbol{R\ (1 - \beta\cos\phi)}} \tag{3.12}$$

c) Figure 3.5 shows the situation in this case. Note the relations given in the caption. Taking $\phi = \phi_1$ we have $\sin\phi_2 = \sin\phi$ and $\cos\phi_2 = -\cos\phi$. Equation (3.12) then gives:

$$\omega_1 = \frac{\beta\ c\ \sin\phi}{R\ (1 - \beta\cos\phi)} \tag{3.13}$$

$$\omega_2 = \frac{\beta\ c\ \sin\phi}{R\ (1 + \beta\cos\phi)} \; . \tag{3.14}$$

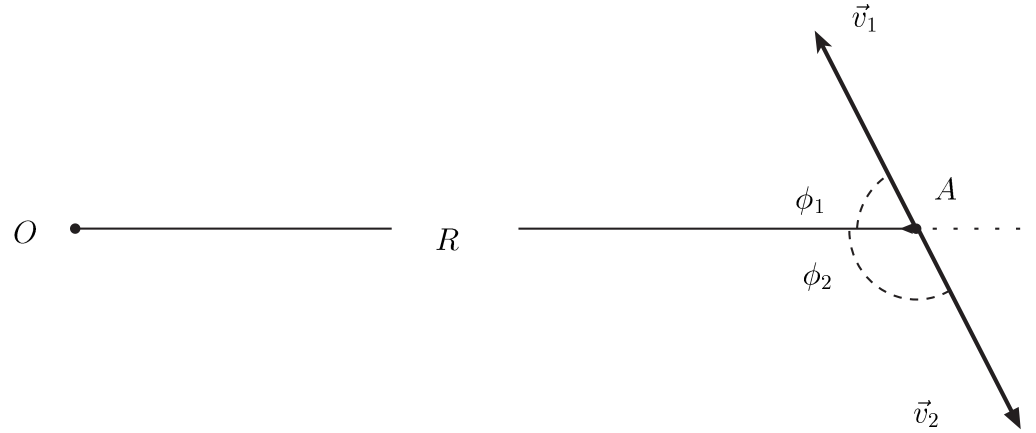


Figure 3.5: *If the two objects have equal speeds but opposite velocities we have* $v_1 = v_2 = v$, $\beta_1 = \beta_2 = \beta$ *and* $\phi_2 = \pi - \phi_1$.

The quantities ω_1, ω_2 and R are given, but β and ϕ are to be determined as stated in the problem text. Simple algebra gives:

$$(1 - \beta\ \cos\phi)\ \omega_1\ \omega_2 \quad = \quad \beta\ c\ \sin\phi\ \omega_2/R \tag{3.15}$$
$$(1 + \beta\ \cos\phi)\ \omega_2\ \omega_1 \quad = \quad \beta\ c\ \sin\phi\ \omega_1/R\ . \tag{3.16}$$

Subtracting (3.15) from (3.16) gives:

$$2\ \beta\ \cos\phi\ \omega_2\ \omega_1 = \beta\ c\ \sin\phi\ (\omega_1 - \omega_2)/R \tag{3.17}$$

$$\tan\phi = \frac{2\ R\ \omega_2\ \omega_1}{c\ \ (\omega_1 - \omega_2)} \tag{3.18}$$

$$\boldsymbol{\phi = \arctan\left(\frac{2\ R\ \omega_2\ \omega_1}{c\ \ (\omega_1 - \omega_2)}\right)}\ . \tag{3.19}$$

Dividing (3.15) by (3.16) gives β in terms of $\cos\phi$ and the known quantities ω_1 and ω_2:

$$\omega_1\ (1 - \beta\ \cos\phi) = \omega_2\ (1 + \beta\ \cos\phi) \tag{3.20}$$

$$\boldsymbol{\beta = \frac{\omega_1 - \omega_2}{\cos\phi\ (\omega_1 + \omega_2)}}\ . \tag{3.21}$$

Inserting the values of ω_1 and ω_2 from part (a) and the given values of R and c we get:

$$\boldsymbol{\phi} \quad = \quad \arctan(2.57) = \mathbf{1.20\ rad = 68.8^\circ \pm 2^\circ} \tag{3.22}$$
$$\boldsymbol{\beta} \quad = \quad \mathbf{0.892 \pm 0.08} \tag{3.23}$$

d) Equation (3.11) shows that the observer will find the apparent transverse speed to be larger than or equal to the speed of light if and only if:

$$\frac{\beta \sin\phi}{1-\beta\cos\phi} \geq 1. \tag{3.24}$$

If $\beta < 1$ condition (3.24) is equivalent to:

$$\begin{aligned} \beta\sin\phi &\geq 1-\beta\cos\phi & (3.25)\\ \beta\left(\sin\phi+\cos\phi\right) &\geq 1 & (3.26)\\ \beta\,\sqrt{2}\left(\sin\phi\cos\frac{\pi}{4}+\cos\phi\sin\frac{\pi}{4}\right) &\geq 1 & (3.27)\\ \sin\left(\phi+\frac{\pi}{4}\right) &\geq \frac{1}{\beta\sqrt{2}} & (3.28)\end{aligned}$$

and hence (3.24) is satisfied if:

$$\beta > \boldsymbol{f(\phi)} = \left(\boldsymbol{\sqrt{2}\sin(\phi+\pi/4)}\right)^{\boldsymbol{-1}}. \tag{3.29}$$

The physically relevant region in the (β,ϕ)-plane is:

$$\boldsymbol{(\beta,\phi)\in[0,1[\times[0,\pi]}\ . \tag{3.30}$$

It is obvious that (3.24) can only be satisfied for $\phi\in[0,\pi/2]$ and (3.28) can only have a solution for ϕ if $\beta\geq 1/\sqrt{2}$.

We therefore take a closer look at the region

$$(\beta,\phi)\ \in\ [2^{-1/2},1[\ \times\ [0,\pi/2] \tag{3.31}$$

The mapping

$$(\beta,\phi)\mapsto\beta\sin\left(\phi+\frac{\pi}{4}\right) \tag{3.32}$$

is continuous in this region. It is therefor sufficient to look at the boundary of the region, defined by the equality sign in (3.28):

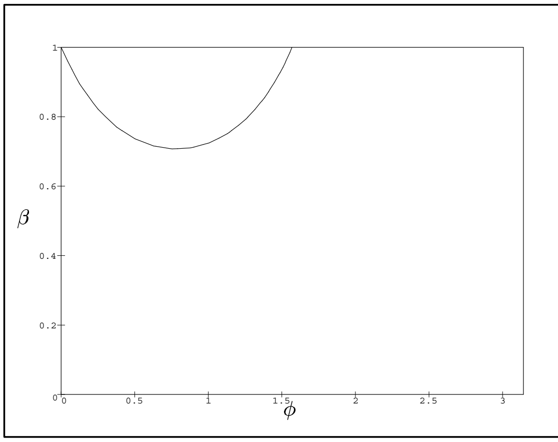
$$\beta\sin\left(\phi+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \tag{3.33}$$

This defines β as a function of ϕ which is shown in Figure 3.6 as the curve bounding the shaded area where $v'_\perp > c$.

e) To find the extrema of $v'_\perp$ as a function of ϕ we differentiate (3.11) and get

$$\frac{d}{d\phi}\left(\frac{v'_\perp}{c}\right)=\frac{\beta(\cos\phi-\beta)}{(1-\beta\cos\phi)^2}\ . \tag{3.34}$$

This is zero for $\phi=\phi_m$ where:



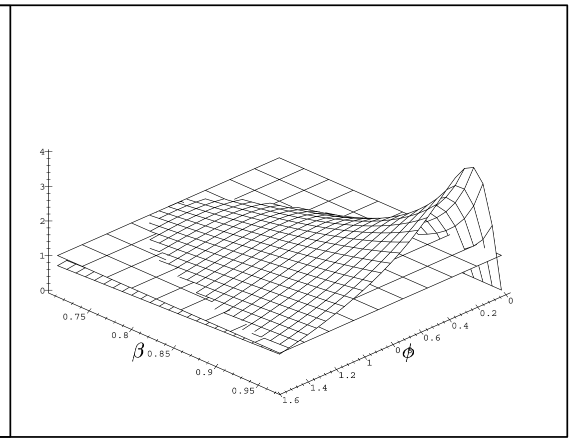


Figure 3.6: *The region between the horizontal line and the curve in the upper left hand corner shows where* $v'_\perp/c > 1$.

Figure 3.7: *The curved surface is* $v'_\perp/c$ *as a function of* β *and* ϕ. *The plane represents the constant function* $\beta = 1$.

$$\cos\phi_m = \beta \ ; \ \phi_m = \arccos\beta \in]0, \pi/2] \tag{3.35}$$

To see that this is indeed a maximum, we differentiate (3.34) again and get:

$$\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right) = -\beta\left(\frac{\sin\phi}{(1-\beta\cos\phi)^2} + 2\frac{\beta\sin\phi(\cos\phi-\beta)}{(1-\beta\cos\phi)^3}\right) \tag{3.36}$$

At the extremum

$$\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right)\bigg|_{\phi_m} = -\frac{\beta\sin\phi_m}{(1-\beta^2)^2} < 0 \tag{3.37}$$

showing that ϕ_m corresponds to a maximum. From (3.11) and (3.35) the maximum apparent transverse speed is given:

$$\boldsymbol{(v'_\perp)_{max} = \frac{\beta c}{\sqrt{1-\beta^2}}} \tag{3.38}$$

From this and (3.35) we see that

$$(v'_\perp)_{max} \underset{\beta\to 1}{\longrightarrow} \infty \ ; \ \phi_m \underset{\beta\to 1}{\longrightarrow} 0 \ . \tag{3.39}$$

Figure 3.7 shows $v'_\perp/c$ as a function of β and ϕ in the region $(\beta, \phi) \in [2^{-1/2}, 1[\times [0, \pi/2]$.

f) We have the equations for relativistic Doppler-shift:

$$\frac{\lambda_{1,2}}{\lambda_0} = \frac{1 \mp \beta \cos\phi}{\sqrt{1-\beta^2}} \tag{3.40}$$

We add them, define an auxiliary ratio ρ and solve for β.

$$\rho := \frac{\lambda_1 + \lambda_2}{2\ \lambda_0} = \frac{1}{\sqrt{1-\beta^2}} \tag{3.41}$$

$$\rho^2\ (1-\beta^2) = 1 \tag{3.42}$$

$$\beta = \sqrt{1 - 1/\rho^2} = \sqrt{1 - \frac{4\ \lambda_0^2}{(\lambda_1+\lambda_2)^2}} \tag{3.43}$$

giving

$$\boldsymbol{\alpha = 4} \tag{3.44}$$

Adding equation (3.43) to the set of equations (3.18) and (3.21) we have three equations which can be solved for the three unknowns β, ϕ and R. For instance, we may calculate β from (3.43), insert that into (3.21), and solve for ϕ. The distance R can then be obtained from (3.18). Thus the measurement of the Doppler-shifted wavelengths turns out to give an estimate of the distance to the source provided that ω_1 and ω_2 are known.

3.3 Grading scheme

Part 1(a)	
Answer i): equation (3.2), ω_1 in the range (16.5-17.5) mas/day	**0.8**
Answer ii): equation (3.3), ω_2 in the range (8.2-9.2) mas/day	**0.8**
Answer iii): equation (3.4), for $v'_{1,\perp}$ in the range (1.13-1.30)c	**0.2**
Answer iv): equation (3.6), for $v'_{2,\perp}$ in the range (0.56-0.70)c	**0.2**
Part 1(b)	
Answer i): $v'_\perp(\beta,\phi)$, equation (3.11)	**2.5**
Answer ii): $\omega(\beta,\phi)$, equation (3.12)	**0.5**
Part 1(c)	
Answer i): $\phi(\omega_1,\omega_2)$, equation (3.19)	**0.3**
Answer ii): $\beta(\omega_1,\omega_2)$, equation (3.21)	**0.3**
Answer iii): ϕ numerical in the range 67° - 71°	**0.2**
Answer iv): β numerical in the range 0.81-0.97	**0.2**
Part 1(d)	
Answer i): Condition $\beta > f(\phi)$, equation (3.29)	**1.0**
Answer ii): Condition on (β,ϕ), graph	**1.0**
Part 1(e)	
Answer: $(v'_\perp)_{max}$, equation (3.38)	**1.0**
Part 1(f)	
Answer: β in terms of λ-s, by α, equation (3.44)	**1.0**