Part 2a

The total energy radiated per second = $4\pi R^2\sigma T^4$, where σ is the Stephan-Boltzmann constant. The energy incident on a unit area on earth per second is;

$$P = \frac{4\pi R^2 \sigma T^4}{4\pi \ell^2} \text{ yielding, } R = \left(P/\sigma T^4\right)^{1/2} \ell \quad (1) \qquad \text{(0.8 pts)}$$

The energy of a photon is $hf=hc/\lambda$. The equivalent mass of a photon is $h/c\lambda$. Conservation of photon energy:

$$\frac{hc}{\lambda_0} - \frac{Gm_0}{R} \cdot \frac{h}{c\lambda_0} = \frac{hc}{\lambda} \qquad \text{(0.8 pts)}$$

yielding

$$R = \frac{Gm_0(\lambda_0 + \Delta\lambda)}{c^2 \Delta\lambda} \quad (2)$$

and (2) yields,

$$m_0 = \frac{c^2 \Delta\lambda \left(P/\sigma T^4\right)^{1/2}}{G(\lambda_0 + \Delta\lambda)} \ell \quad (3) \qquad \text{(0.2 pts)}$$

The stars are rotating around the center of mass with equal angular speeds:

$\omega = (2\pi/2\tau) = \pi/\tau$ (4) (0.2 pts)

The equilibrium conditions for the stars are;

$$\frac{GMm_0}{(r_1 + r_2)^2} = m_0 r_1 \omega^2 = M r_2 \omega^2 \quad (5) \qquad \text{(0.8 pts)}$$

with

$$r_1 = \ell \frac{\Delta\theta}{2}, \; r_2 = \ell \frac{\Delta\phi}{2} \quad (6) \qquad \text{(0.4 pts)}$$

Substituting (3), (4) and (6) into (5) yields

$$\ell = \left(\frac{8c^2 \Delta\lambda \left(P/\sigma T^4\right)^{1/2}}{\Delta\phi (\pi/\tau)^2 (\lambda_0 + \Delta\lambda)(\Delta\theta + \Delta\phi)^2} \right)^{1/2}. \qquad \text{(0.8 pts)}$$

Part 2b

Conservation of angular momentum for the ordinary star;

$$mr^2 \omega = m_0 r_0^2 \omega_0 \quad (7) \qquad \text{(0.6 pts.)}$$

Conservation of angular momentum for dm:

$$r^2 \omega \, dm = r_f^2 \omega_f \, dm \quad (8) \qquad \text{(0.6 pts)}$$

where ω_f is the angular velocity of the ring. Equilibrium in the original state yields,

$$\omega_0 = \left(\frac{GM}{r_0^3} \right)^{1/2} \quad (9) \qquad \text{(0.8 pts)}$$

and (7), (8) and (9) give,

$$\omega = \frac{m_0 r_0}{m r^2}\left(\frac{GM}{r_0}\right)^{1/2}, \quad \omega_f = \frac{m_0 r_0}{m r_f^2}\left(\frac{GM}{r_0}\right)^{1/2} \quad (10)$$ (0.4 pts)

Conservation of energy for dm;

$$\frac{1}{2} dm\left(v_0^2 + r^2\omega^2\right) - \frac{GM\,\mathrm{dm}}{\mathrm{r}} = \frac{1}{2} dm\, r_f{}^2 \omega_f{}^2 - \frac{GM\,\mathrm{dm}}{\mathrm{r_f}} \quad (11)$$ (1.2 pts)

Substituting (10);

$$v_0^2 + \frac{m_0^2 r_0 GM}{m^2}\left(\frac{1}{r^2} - \frac{1}{r_f^2}\right) - 2GM\left(\frac{1}{r} - \frac{1}{r_f}\right) = 0 \quad (12)$$

Since $r_0 >> r_f$, if $r > r_0$, r^{-1} and r^{-2} terms can be neglected. Hence,

$$r_f = \frac{GM}{v_0^2}\left(\left(1 + \frac{m_0^2 r_0 v_0^2}{GMm^2}\right)^{1/2} - 1\right).$$ (0.8 pts)

To show that $r>r_0$ change in the linear momentum of the ordinary star in its reference frame:

$$-\frac{GMm}{r^2} + mr\omega^2 - m\frac{dv_r}{dt} = -v_0\frac{dm_{gas}}{dt} \quad (13)$$ (0.8 pts)

and (13) implies the existence of an outward force initially and hence r starts growing. Using (7) one can write

$$mr\omega^2 = \frac{m_0^2 r_0^4 \omega_0^2}{mr^3} .$$

Hence, $\dfrac{Gravitational\ force}{Centrifugal\ force}\, \alpha\, \mathrm{m}^2 r$. (0.4 pts)

where m is definitely decreasing. If r starts decreasing at some time also, this ratio starts decreasing, which is a contradiction.

So $r>r_0$. (0.4 pts)