Solution- Theoretical Question 1

A Swing with a Falling Weight

Part A

(a) Since the length of the string $L = s + R\theta$ is constant, its rate of change must be zero. Hence we have

$$\dot{s} + R\dot{\theta} = 0 \qquad \text{(A1)*[1]}$$

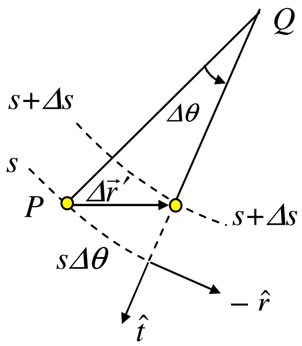
(b) Relative to O, Q moves on a circle of radius R with angular velocity $\dot{\theta}$, so

$$\vec{v}_Q = R\dot{\theta}\hat{t} = -\dot{s}\hat{t} \qquad \text{(A2)*}$$

(c) Refer to Fig. A1. Relative to Q, the displacement of P in a time interval Δt is $\Delta\vec{r}\,' = (s\Delta\theta)(-\hat{r}) + (\Delta s)\hat{t} = [(s\dot{\theta})(-\hat{r}) + \dot{s}\hat{t}]\Delta t$. It follows

$$\vec{v}\,' = -s\dot{\theta}\hat{r} + \dot{s}\hat{t} \qquad \text{(A3)*}$$

Figure A1



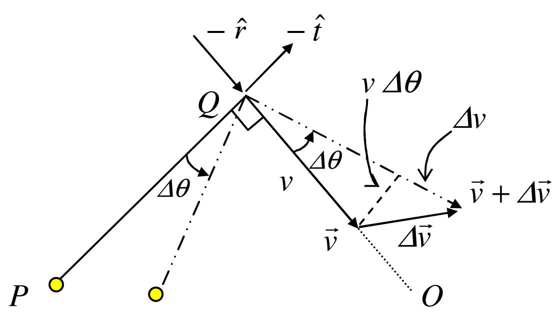
(d) The velocity of the particle relative to O is the sum of the two relative velocities given in Eqs. (A2) and (A3) so that

$$\vec{v} = \vec{v}\,' + \vec{v}_Q = (-s\dot{\theta}\hat{r} + \dot{s}\hat{t}) + R\dot{\theta}\hat{t} = -s\dot{\theta}\hat{r} \qquad \text{(A4)*}$$

(e) Refer to Fig. A2. The $(-\hat{t})$-component of the velocity change $\Delta\vec{v}$ is given by $(-\hat{t})\cdot\Delta\vec{v} = v\Delta\theta = v\dot{\theta}\Delta t$. Therefore, the $\hat{t}$ -component of the acceleration $\vec{a} = \Delta\vec{v}/\Delta t$ is given by $\hat{t}\cdot\hat{a} = -v\dot{\theta}$. Since the speed v of the particle is $s\dot{\theta}$ according to Eq. (A4), we see that the $\hat{t}$ -component of the particle's acceleration at P is given by

$$\vec{a}\cdot\hat{t} = -v\dot{\theta} = -(s\dot{\theta})\dot{\theta} = -s\dot{\theta}^2 \qquad \text{(A5)*}$$

Figure A2



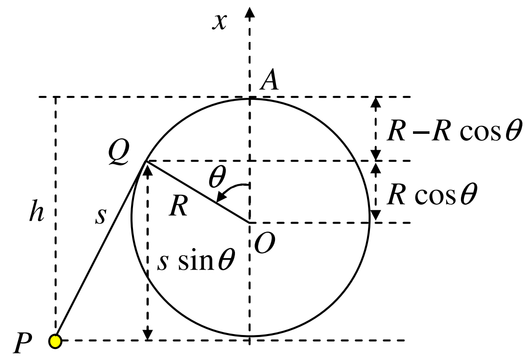
[1] An equation marked with an asterisk contains answer to the problem.

Note that, from Fig. A2, the radial component of the acceleration may also be obtained as $\vec{a}\cdot\hat{r} = -dv/dt = -d(s\dot{\theta})/dt$.

(f) Refer to Fig. A3. The gravitational potential energy of the particle is given by $U = -mgh$. It may be expressed in terms of s and θ as

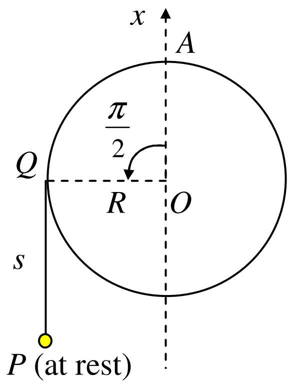
$$U(\theta) = -mg[R(1-\cos\theta) + s\sin\theta] \qquad \text{(A6)*}$$

Figure A3



(g) At the lowest point of its trajectory, the particle's gravitational potential energy U must assume its minimum value U_m. If the particle's mechanical energy E were equal to U_m, its kinetic energy would be zero. The particle would then remain stationary and be in the static equilibrium state shown in Fig. A4. Thus, the potential energy reaches its minimum value when $\theta = \pi/2$ or $s = L - \pi R/2$.

Figure A4



From Fig. A4 or Eq. (A6), the minimum potential energy is then

$$U_m = U(\frac{\pi}{2}) = -mg[R + L - (\pi R/2)] . \qquad \text{(A7)}$$

Initially, the total mechanical energy E is 0. Since E is conserved, the speed v_m of the particle at the lowest point of its trajectory must satisfy

$$E = 0 = \frac{1}{2}mv_m^2 + U_m . \qquad \text{(A8)}$$

From Eqs. (A7) and (A8), we obtain

$$v_m = \sqrt{-2U_m/m} = \sqrt{2g[R + (L - \pi R/2)]} . \qquad \text{(A9)*}$$

Part B

(h) From Eq. (A6), the total mechanical energy of the particle may be written as

$$E = 0 = \frac{1}{2}mv^2 + U(\theta) = \frac{1}{2}mv^2 - mg[R(1-\cos\theta) + s\sin\theta] \qquad \text{(B1)}$$

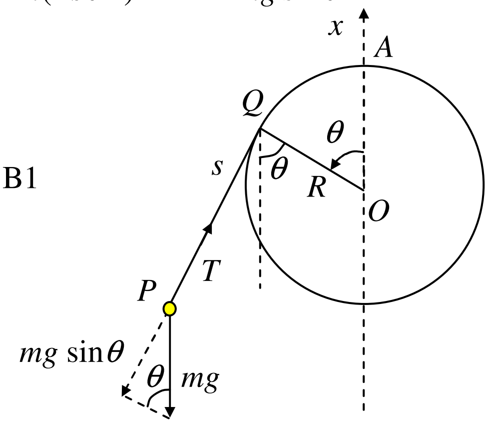
From Eq. (A4), the speed v is equal to $s\dot{\theta}$. Therefore, Eq. (B1) implies

$$v^2 = (s\dot{\theta})^2 = 2g[R(1-\cos\theta) + s\sin\theta] \qquad \text{(B2)}$$

Let T be the tension in the string. Then, as Fig. B1 shows, the $\hat{t}$-component of the net force on the particle is $-T + mg\sin\theta$. From Eq. (A5), the tangential acceleration of the particle is $(-s\dot{\theta}^2)$. Thus, by Newton's second law, we have

$$m(-s\dot{\theta}^2) = -T + mg\sin\theta \qquad \text{(B3)}$$

Figure B1

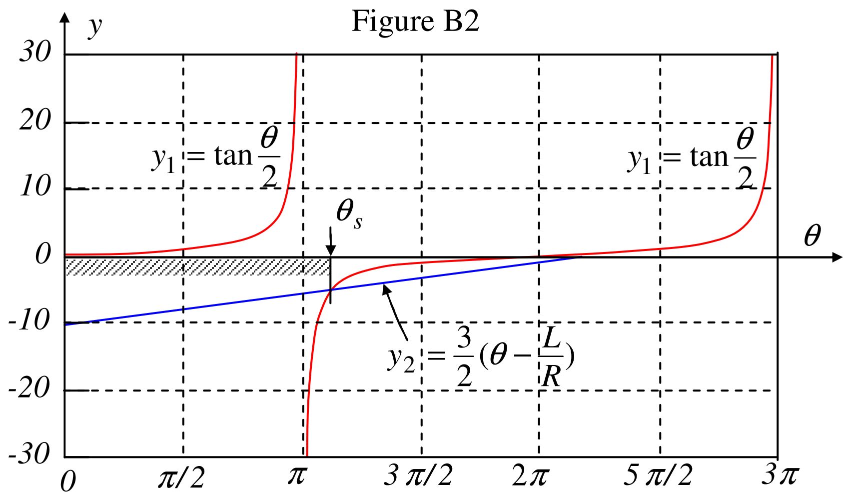


According to the last two equations, the tension may be expressed as

$$\begin{aligned} T &= m(s\dot{\theta}^2 + g\sin\theta) = \frac{mg}{s}[2R(1-\cos\theta) + 3s\sin\theta] \\ &= \frac{2mgR}{s}\left[\tan\frac{\theta}{2} - \frac{3}{2}\left(\theta - \frac{L}{R}\right)\right](\sin\theta) \qquad \text{(B4)} \\ &= \frac{2mgR}{s}(y_1 - y_2)(\sin\theta) \end{aligned}$$

The functions $y_1 = \tan(\theta/2)$ and $y_2 = 3(\theta - L/R)/2$ are plotted in Fig B2.

Figure B2



From Eq. (B4) and Fig. B2, we obtain the result shown in Table B1. The angle at which $y_2 = y_1$ is called θ_s ($\pi < \theta_s < 2\pi$) and is given by

$$\frac{3}{2}(\theta_s - \frac{L}{R}) = \tan\frac{\theta_s}{2} \qquad \text{(B5)}$$

or, equivalently, by

$$\frac{L}{R} = \theta_s - \frac{2}{3}\tan\frac{\theta_s}{2} \qquad \text{(B6)}$$

Since the ratio L/R is known to be given by

$$\frac{L}{R} = \frac{9\pi}{8} + \frac{2}{3}\cot\frac{\pi}{16} = (\pi + \frac{\pi}{8}) - \frac{2}{3}\tan\frac{1}{2}(\pi + \frac{\pi}{8}) \qquad \text{(B7)}$$

one can readily see from the last two equations that $\theta_s = 9\pi/8$.

Table B1

	$(y_1 - y_2)$	$\sin\theta$	tension T
$0 < \theta < \pi$	positive	positive	positive
$\theta = \pi$	$+\infty$	0	positive
$\pi < \theta < \theta_s$	negative	negative	positive
$\theta = \theta_s$	zero	negative	zero
$\theta_s < \theta < 2\pi$	positive	negative	negative

Table B1 shows that the tension T must be positive (or the string must be taut and straight) in the angular range $0 < \theta < \theta_s$. Once θ reaches θ_s, the tension T becomes zero and the part of the string not in contact with the rod will not be straight afterwards. The shortest possible value $s_{\min}$ for the length s of the line segment QP therefore occurs at $\theta = \theta_s$ and is given by

$$s_{\min} = L - R\theta_s = R(\frac{9\pi}{8} + \frac{2}{3}\cot\frac{\pi}{16} - \frac{9\pi}{8}) = \frac{2R}{3}\cot\frac{\pi}{16} = 3.352R \qquad \text{(B8)}$$

When $\theta = \theta_s$, we have $T = 0$ and Eqs. (B2) and (B3) then leads to $v^2 = -gs\sin\theta$. Hence the speed v_s is

$$v_s = \sqrt{-gs_{\min}\sin\theta_s} = \sqrt{\frac{2gR}{3}\cot\frac{\pi}{16}\sin\frac{\pi}{8}} = \sqrt{\frac{4gR}{3}}\cos\frac{\pi}{16} \qquad \text{(B9)*}$$
$$= 1.133\sqrt{gR}$$

(i) When $\theta \geq \theta_s$, the particle moves like a projectile under gravity. As shown in Fig. B3, it is projected with an initial speed v_s from the position $P = (x_s, y_s)$ in a direction making an angle $\phi = (3\pi/2 - \theta_s)$ with the y-axis.

The speed v_H of the particle at the highest point of its parabolic trajectory is equal to the y-component of its initial velocity when projected. Thus,

$$v_H = v_s\sin(\theta_s - \pi) = \sqrt{\frac{4gR}{3}}\cos\frac{\pi}{16}\sin\frac{\pi}{8} = 0.4334\sqrt{gR} \qquad \text{(B10)*}$$

The horizontal distance H traveled by the particle from point P to the point of maximum height is

$$H = \frac{v_s^2\sin 2(\theta_s - \pi)}{2g} = \frac{v_s^2}{2g}\sin\frac{9\pi}{4} = 0.4535R \qquad \text{(B11)}$$

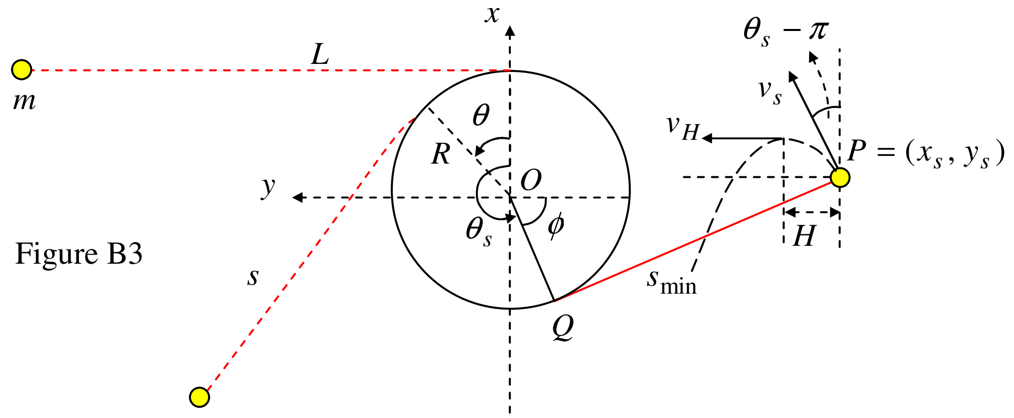


Figure B3

The coordinates of the particle when $\theta = \theta_s$ are given by

$$x_s = R\cos\theta_s - s_{\min}\sin\theta_s = -R\cos\frac{\pi}{8} + s_{\min}\sin\frac{\pi}{8} = 0.358R \qquad \text{(B12)}$$

$$y_s = R\sin\theta_s + s_{\min}\cos\theta_s = -R\sin\frac{\pi}{8} - s_{\min}\cos\frac{\pi}{8} = -3.478R \qquad \text{(B13)}$$

Evidently, we have $|y_s| > (R + H)$. Therefore the particle can indeed reach its maximum height without striking the surface of the rod.

Part C

(j) Assume the weight is initially lower than O by h as shown in Fig. C1.

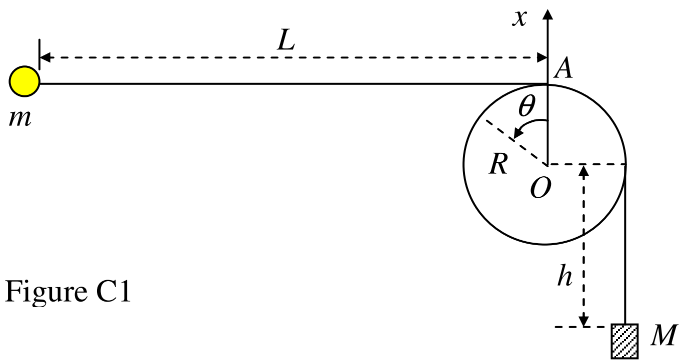


Figure C1

When the weight has fallen a distance D and stopped, the law of conservation of total mechanical energy as applied to the particle-weight pair as a system leads to

$$-Mgh = E' - Mg(h+D) \tag{C1}$$

where E' is the *total mechanical energy of the particle* when the weight has stopped. It follows

$$E' = MgD \tag{C2}$$

Let Λ be the total length of the string. Then, its value at $\theta = 0$ must be the same as at any other angular displacement θ. Thus we must have

$$\Lambda = L + \frac{\pi}{2}R + h = s + R(\theta + \frac{\pi}{2}) + (h+D) \tag{C3}$$

Noting that $D = \alpha L$ and introducing $\ell = L - D$, we may write

$$\ell = L - D = (1-\alpha)L \tag{C4}$$

From the last two equations, we obtain

$$s = L - D - R\theta = \ell - R\theta \tag{C5}$$

After the weight has stopped, the total mechanical energy of the particle must be conserved. According to Eq. (C2), we now have, instead of Eq. (B1), the following equation:

$$E' = MgD = \frac{1}{2}mv^2 - mg[R(1-\cos\theta) + s\sin\theta] \tag{C6}$$

The square of the particle's speed is accordingly given by

$$v^2 = (s\dot{\theta})^2 = \frac{2MgD}{m} + 2gR[(1-\cos\theta) + \frac{s}{R}\sin\theta] \quad . \tag{C7}$$

Since Eq. (B3) stills applies, the tension T of the string is given by

$$-T + mg\sin\theta = m(-s\dot{\theta}^2) \tag{C8}$$

From the last two equations, it follows

$$\begin{aligned} T &= m(s\dot{\theta}^2 + g\sin\theta) \\ &= \frac{mg}{s}[\frac{2M}{m}D + 2R(1-\cos\theta) + 3s\sin\theta] \\ &= \frac{2mgR}{s}[\frac{MD}{mR} + (1-\cos\theta) + \frac{3}{2}(\frac{\ell}{R} - \theta)\sin\theta] \end{aligned} \tag{C9}$$

where Eq. (C5) has been used to obtain the last equality.

We now introduce the function

$$f(\theta) = 1 - \cos\theta + \frac{3}{2}(\frac{\ell}{R} - \theta)\sin\theta \tag{C10}$$

From the fact $\ell = (L-D) >> R$, we may write

$$f(\theta) \approx 1 + \frac{3}{2}\frac{\ell}{R}\sin\theta - \cos\theta = 1 + A\sin(\theta - \phi) \qquad \text{(C11)}$$

where we have introduced

$$A = \sqrt{1 + (\frac{3}{2}\frac{\ell}{R})^2}\ , \qquad \phi = \tan^{-1}\frac{\frac{3\ell}{2R}}{\sqrt{1 + (\frac{3\ell}{2R})^2}} \qquad \text{(C12)}$$

From Eq. (C11), the minimum value of $f(\theta)$ is seen to be given by

$$f_{\min} = 1 - A = 1 - \sqrt{1 + (\frac{3}{2}\frac{\ell}{R})^2} \qquad \text{(C13)}$$

Since the tension T remains nonnegative as the particle swings around the rod, we have from Eq. (C9) the inequality

$$\frac{MD}{mR} + f_{\min} = \frac{M(L-\ell)}{mR} + 1 - \sqrt{1 + (\frac{3\ell}{2R})^2} \geq 0 \qquad \text{(C14)}$$

or

$$(\frac{ML}{mR}) + 1 \ \geq \ (\frac{M\ell}{mR}) + \sqrt{1 + (\frac{3\ell}{2R})^2} \approx (\frac{M\ell}{mR}) + (\frac{3\ell}{2R}) \qquad \text{(C15)}$$

From Eq. (C4), Eq. (C15) may be written as

$$(\frac{ML}{mR}) + 1 \ \geq \ [(\frac{ML}{mR}) + (\frac{3L}{2R})](1-\alpha) \qquad \text{(C16)}$$

Neglecting terms of the order (R/L) or higher, the last inequality leads to

$$\alpha \ \geq \ 1 - \frac{(\frac{ML}{mR}) + 1}{(\frac{ML}{mR}) + (\frac{3L}{2R})} = \frac{(\frac{3L}{2R}) - 1}{(\frac{ML}{mR}) + (\frac{3L}{2R})} = \frac{1 - \frac{2R}{3L}}{\frac{2M}{3m} + 1} \approx \frac{1}{1 + \frac{2M}{3m}} \qquad \text{(C17)}$$

The critical value for the ratio D/L is therefore

$$\alpha_c = \frac{1}{(1 + \frac{2M}{3m})} \qquad \text{(C18)*}$$